

Math 25b: Honors Linear Algebra and Real Analysis II

Homework Assignment #3 (14 February 2014):

Metric Topology III: uniform continuity, compactness, and completeness

“There goes the open ball.”

—Mangled punchline of a hoary “ $\epsilon \rightarrow 0$ ” joke

Due to the Feb. 17 holiday, this problem set covers only two class meetings; therefore it consists of only six problems.

N.B. The sections for particular CA’s are not sequential! This was done to group related questions; sorry for the inconvenience.

Uniform vs. non-uniform continuity:

1. [**Sadik**] Let P be a polynomial with real coefficients, considered as a function from \mathbf{R} to \mathbf{R} (with the usual metric on both the source and target \mathbf{R}). We have seen in the previous problem set that P is continuous.
 - i) Prove that if $\deg(P) \leq 1$ then P is uniformly continuous.
 - ii) Prove that if $\deg(P) > 1$ then P is not uniformly continuous. [Hint: show that if any function $f : \mathbf{R} \rightarrow \mathbf{R}$ is uniformly continuous then there exists $\delta > 0$ such that $|f(n\delta) - f(0)| \leq n$ for each $n = 1, 2, 3, \dots$]
2. [**Sadik**] Let m and n be any positive integers. Prove that every linear transformation from \mathbf{R}^m to \mathbf{R}^n is *uniformly* continuous. [You may use either the sup metric or the Euclidean metric. You’ll first have to prove the function is continuous at $\mathbf{0}$; use the matrix form of a linear transformation.]

We noted that once we’ve shown that the product of two compact spaces is compact then by induction the same is true for any finite product of compact spaces. But (using the sup metric) it doesn’t work for a countably infinite product...

3. [**Tudor**] Let X be the product of a countable infinite of copies of the unit interval $[0, 1]$, or equivalently of sequences $p = (p_1, p_2, p_3, \dots)$ with each $p_n \in [0, 1]$. Then $d(p, q) = \sup_n |p_n - q_n|$ defines a metric on X . Prove that X is not totally bounded (first find a sequence x_1, x_2, \dots with each $x_m \in X$ such that $d(x_m, x_n) = 1$ for all $m \neq n$), and therefore not compact.

A non-convergent Cauchy sequence in a function space:

4. [**Nat**] As in Problem 6 of the previous problem set, define a metric

$$d_1(f, g) := \int_{-1}^1 |f(x) - g(x)| dx$$

on the space $\mathcal{C}([-1, 1], \mathbf{R})$ of (bounded) continuous functions $f : [-1, 1] \rightarrow \mathbf{R}$.¹ Let f_n be the function given by

$$f_n(x) = \begin{cases} nx, & \text{if } |nx| \leq 1; \\ 1, & \text{if } nx \geq 1; \\ -1, & \text{if } nx \leq -1. \end{cases}$$

¹We previously used $[0, 1]$ but for the present purpose $[-1, 1]$ is more convenient; and we do not need complex-valued functions so we content ourselves with $\mathcal{C}([-1, 1], \mathbf{R})$ rather than $\mathcal{C}([-1, 1], \mathbf{C})$.

(Note that when $x = \pm 1/n$ two clauses apply but both give ± 1 , so the definition is consistent.)

- i) Show that each f_n is continuous, and thus contained in $\mathcal{C}([-1, 1], \mathbf{R})$.
- ii) Calculate that $d_1(f_m, f_n) = |1/m - 1/n|$, and deduce that $\{f_n\}$ is a Cauchy sequence in $\mathcal{C}([-1, 1], \mathbf{C})$.

Intuitively the reason that this sequence has no limit in $\mathcal{C}([-1, 1], \mathbf{C})$ is that the limit should be the function that's -1 for $x < 0$ and $+1$ for $x > 0$, and this function is not continuous at $x = 0$. Here's a proof:

5. **[Nat]** Let f be any continuous function from $[-1, 1]$ to \mathbf{R} . There exists δ such that $|x| \leq \delta \Rightarrow |f(x) - f(0)| < \frac{1}{4}$ (why?). Show that once $1/n < \delta/2$ the distance from f to f_n is at least $\delta/4$ (due either to the integral over $-\delta \leq x \leq -\delta/2$ or to the integral over $\delta/2 \leq x \leq \delta$). Explain why this shows that f is not a limit of $\{f_n\}$, and thus that $\{f_n\}$ is not convergent in $\mathcal{C}([-1, 1], \mathbf{R})$.

An application of total boundedness in \mathbf{R} and \mathbf{R}^2 :

6. **[Tudor]** i) Fix a real number x , and let for each $n = 1, 2, 3, \dots$ let p_n be the fractional part of nx . (The fractional part of a real number r is the unique real number $f \in [0, 1)$ such that $r - f$ is an integer. For example, if $x = e$ then $p_1 = e - 2 = 0.71828\dots$, $p_2 = 2e - 5 = 0.43656\dots$, and $p_{100} = 100e - 271 = 0.82818\dots$; if some nx is an integer then we take $p_n = 0$.) Thus $\{p_n\}$ a sequence in the semiopen interval $[0, 1)$. This interval, regarded as a subspace of \mathbf{R} (with the usual metric structure), is totally bounded. Therefore the sequence $\{p_n\}$ has a Cauchy subsequence. Use this to deduce that for every ϵ there exist integers a, b with $a > 0$ such that $|ax - b| < \epsilon$.
- ii) Now let x, y be real numbers and let p_n be the point in \mathbf{R}^2 whose coordinates are the fractional parts of nx and ny . This is a sequence in $[0, 1)^2$, a subspace of \mathbf{R}^2 (use the standard metric on \mathbf{R} and the sup metric on \mathbf{R}^2). Use the total boundedness of $[0, 1)^2$ to deduce that for every ϵ there exist integers a, b, c such that $|ax - b| < \epsilon$ and $|ay - c| < \epsilon$.

Yes, the same argument shows that for real x_1, \dots, x_k and any ϵ there are integers $a > 0$ and b_1, \dots, b_k such that $|ax_i - b_i| < \epsilon$ for each $i = 1, 2, \dots, k$.

The problem set is due Friday, February 21, at 5PM.