**Math 259: Introduction to Analytic Number Theory**

Some more about modular forms

**The coefficients of the $q$-expansions of modular forms and functions**

Concerning "Theorem 5 (Hecke)" of [Serre 1973, VII (p.94)]: We use Poincaré series and estimates on Kloosterman sums to show that the exponent $k$ in $a_n = O(n^k)$ can be improved to $k - \frac{1}{4} + o(1)$. This method, while neither as deep nor as powerful as Deligne's derivation of the correct $k - \frac{1}{2} + o(1)$, is more generally applicable.

The coefficients of the $q$-series for $j(z)$ grow much more rapidly. Let us write $j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n$, adopting the notation in [Serre 1973, p.90]. An easy upper bound follows from the positivity of the $c(n)$:

**Lemma.** $c_n \ll \exp(4\pi\sqrt{n})$.

*Proof*: All the $c(n)$ are positive by the formula

$$j(z) = E_4^3/\Delta = q^{-1}E_4^3 \prod_{n=1}^{\infty}(1-q^n)^{-24}.$$

Therefore if we take $z = iy$ then $q = e^{-2\pi y} > 0$ and we find

$$c(n) = q^{-n} \cdot c(n)q^n < e^{2\pi n y} j(iy). \tag{1}$$

If $y$ is bounded above, say $y < y_0$, then

$$j(iy) = j(i/y) = e^{2\pi/y} + O(1),$$

because $\exp(-2\pi/y) < \exp(2\pi y_0)$ and the series $744 + \sum_{n=1}^{\infty} c(n)e^{-2\pi n/y_0}$ for $j(i/y_0) - \exp(2\pi/y_0)$ converges. Thus (1) yields $c(n) \ll \exp(2\pi(ny + y^{-1}))$. Taking $y = n^{-1/2}$, we deduce $c(n) \ll \exp(4\pi\sqrt{n}\,)$, as claimed. $\Box$

It turns out that this bound exceeds the actual growth of $c(n)$ by a factor of only $O(\sqrt{n}\,)$. Asymptotic formulas for $c(n)$ and the coefficients of other modular functions with poles at $i\infty$ can be obtained by the "stationary-phase method", which for $c(n)$ amounts to estimating the integrand in the Fourier integral

$$c(n) = \int_{-1/2}^{1/2} j(x + in^{-1/2})\,dx$$

near the point $x = 0$ at which it is maximal. The Hardy-Ramanujan method and Rademacher's refinement of it, originally applied to the partition function (see below), also yields asymptotic expansions and convergent series respectively for $c(n)$ and the coefficients of similar $q$-expansions.

**Product formulas for modular forms**

**The product formula for $\Delta$.** Concerning "Theorem 6 (Jacobi)" of [Serre 1973, VII (p.95)]: A more direct and arguably simpler way to prove the key relation $G_1(z) - G(z) = 2\pi i/z$ between

$$G_1(z) = \sum_n \sum_m{}' \frac{1}{(m+nz)^2} \quad \text{and} \quad G(z) = \sum_m \sum_n{}' \frac{1}{(m+nz)^2}$$

is as follows. By the $q$-expansions, both series converge. Therefore

$$G_1(z) - G(z) = \lim_{N\to\infty} \left( \sum_{n=-N}^{N} \sum_m{}' \frac{1}{(m+nz)^2} - \sum_{m=-N}^{N} \sum_n{}' \frac{1}{(m+nz)^2} \right).$$

Now rewrite the limit as

$$\lim_{N\to\infty} \left[ \sum_{|m|>N} \sum_{|n|\le N} - \sum_{|m|\le N} \sum_{|n|>N} \right] \frac{1}{(m+nz)^2},$$

and note that $1/(m+nz)^2 = N^{-2}/\big((m+nz)/N\big)^2$ to recognize the sums as Riemann sums for the double integrals

$$\int_{|r|\ge 1} \int_{|s|\le 1} \frac{dr\,ds}{(r+sz)^2}, \qquad \int_{|r|\le 1} \int_{|s|\ge 1} \frac{dr\,ds}{(r+sz)^2}.$$

But these integrals are elementary:

$$\int_{|r|\ge 1} \int_{|s|\le 1} \frac{dr\,ds}{(r+sz)^2} = \int_{|r|\ge 1} \frac{2\,dr}{r^2 - z^2}\,, \tag{2}$$

and similarly

$$\int_{|r|\le 1} \int_{|s|\ge 1} \frac{dr\,ds}{(r+sz)^2} = \int_{|r|\le 1} \frac{-2\,dr}{r^2 - z^2}\,. \tag{3}$$

We could now evaluate the integrals directly using partial fractions, but an even cleaner conclusion is to combine (2) and (3), writing their difference as $2\int_{r=-\infty}^{\infty} dr/(r^2 - z^2)$. This is easily evaluated using contour integration. Since $z$ has positive imaginary part, the unique pole of the integrand in the upper half-plane is at $r = z$, with residue $1/2z$. Therefore the integral equals $2\pi i/z$, and we are done.

**The modular form $\eta$ of weight one-half.** Jacobi's product formula

$$\Delta(z) = q \prod_{q=1}^{\infty} (1 - q^n)^{24} \tag{4}$$

suggests that we consider the 24th root

$$\eta(z) = q^{1/24} \prod_{q=1}^{\infty} (1 - q^n) \tag{5}$$

of $\Delta$ as a "modular form of weight 1/2" for the full modular group $\mathrm{PSL}_2(\mathbf{Z})$. (By $q^{1/24}$ we mean the branch $e^{2\pi i z/24}$ of the 24th root of $q$.) That is, for any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ we have

$$\eta\Big(\frac{az+b}{cz+d}\Big) = \epsilon_g (cz+d)^{1/2} \eta(z) \tag{6}$$

for some 24th root of unity $\epsilon_g$. It turns out that $\epsilon_g$ depends in a complicated but predictable way on $g \bmod 24$. One can similarly define modular forms of half-integral weight for other congruence subgroups of $\mathrm{PSL}_2(\mathbf{Z})$. It turns out that this does not generalize nicely beyond half-integers; for instance, while fractional powers of $\eta$ are analytic functions on the upper half-plane that satisfy transformation formulas analogous to (6), the factors corresponding to $\epsilon_g$ do not depend on $g \bmod N$ for any $N$. Also, unlike all the modular forms and functions that we will study, fractional powers of $\eta$ do not have $q$-expansions with (algebraic) integer coefficients, which suggests that they lack the number-theoretic significance of modular forms and functions of integral and half-integral weight.

**The $q$-series for $\eta$, and the partition function.** The trick that Serre uses to finish the proof of (4) can be applied to prove other identities involving modular forms. One example is the following proof of Euler's celebrated "pentagonal-number" formula[1]

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^n) &= \sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m+1)/2} \qquad (7)\\ &= 1-q-q^2+q^5+q^7-q^{12}-++-\cdots\end{aligned}$$

for the product that appears in the formula for $\eta(z)$. Completing the square, we find that

$$\frac{1}{2}m(3m+1)+\frac{1}{24}=3\,\frac{(6m+1)^2}{24}=3\,\frac{(-6m-1)^2}{24},$$

and thus that

$$2q^{1/24}\sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m+1)/2}=\sum_{m=-\infty}^{\infty}(-1)^m\left(q^{(6m+1)^2/24}+q^{(6m-1)^2/24}\right).$$

Taking $k=6m\pm 1$, we may rewrite the sum as

$$2q^{1/24}\sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m+1)/2}=\sum_{k=-\infty}^{\infty}\chi(m)q^{k^2/24}$$

where $\chi$ is the Dirichlet character mod 12 given by $\chi(6m\pm 1)=(-1)^m$ (that is, $\chi=\chi_3\chi_4$). We have thus recognized $2q^{1/24}\sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m+1)/2}$ as a twisted theta function, namely $\theta_\chi(z/12i)$ in our notation. The functional equation for such twisted theta functions then yields

$$\theta_\chi(z/12i)=(z/i)^{1/2}\theta_\chi((-1/z)/12i).$$

It follows that $q^{1/24}\sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m+1)/2}$ transforms under $\mathrm{PSL}_2(\mathbf{Z})$ in the same way that $q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ does. Again we use the maximum principle

[1] The exponents $m(3m+1)/2$ are known as "pentagonal numbers", at least for $m>0$; these continue the progression that begins with triangular numbers $m(m-1)/2$ and squares $m^2$.

to deduce that the two functions are proportional, and compare leading terms to conclude that they are equal.

[Further examples of this approach to proving identities between $q$-series can be found in the Exercises. In general $\theta_\chi(z/qi)$ is a modular form of weight 1/2 for some congruence subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ depending on $q$. The pentagonal-number theorem, as well as some of the identities in the Exercises, can also be obtained as special cases of Jacobi's triple-product identity

$$\sum_{m=-\infty}^{\infty} u^m q^{m^2} = \prod_{n=1}^{\infty} (1+uq^{2n-1})(1+u^{-1}q^{2n-1})(1-q^{2n}).$$

For example, (7) is obtained by substituting $(q^{3/2}, -q^{1/2})$ for $(q,u)$.]

A remarkable consequence of (7) is Euler's recurrence for the *partition function* $p(r)$. This is the number of ways to write an integer $r \geq 0$ as the sum of positive integers up to rearrangement of the summands, or equivalently as $\sum_{n=1}^{\infty} a_n n$ (with $a_n$ being the number of summands that equal $n$). We readily obtain the generating function

$$\sum_{r=0}^{\infty} p(r)q^r = \prod_{n=1}^{\infty}(1+q^n+q^{2n}+q^{3n}+\ldots) = \prod_{n=1}^{\infty}(1-q^n)^{-1}$$
$$= 1+q+2q^2+3q^3+5q^4+7q^5+11q^6+15q^7+\cdots.$$

By (7) this yields $\sum_{m=-\infty}^{\infty}(-1)^m p(r-\frac{1}{2}m(3m+1)) = 0$ for all integers $m > 0$. (The sum is finite because $p(r)=0$ for $r<0$.) Therefore

$$p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+--+\cdots.$$

**Weighted theta functions**

Suppose $\Gamma$ is a lattice in the $n$-dimensional real inner product space $V$, and $\Gamma'$ the dual lattice $\{\mathbf{x} \in V : \forall \mathbf{y} \in \Gamma, (\mathbf{x},\mathbf{y}) \in \mathbf{Z}\}$. We have seen already that the Poisson summation formula applied to the function $\exp(-\pi t|\mathbf{x}|^2)$ yields a functional equation connecting the theta functions of $\Gamma$ and $\Gamma'$, which Serre [1973, VII, Prop.16, p.107] writes in the form

$$\Theta_\Gamma(t) = t^{-n/2} v^{-1} \Theta_{\Gamma'}(t^{-1})$$

with $v = \mathrm{Vol}(V/\Gamma)$ the covolume and

$$\Theta_\Gamma(t) := \sum_{\mathbf{x}\in\Gamma} \exp(-\pi t |\mathbf{x}|^2).$$

Using the functional equation, Serre then shows that when $\Gamma$ is a unimodular even lattice the function $\theta_\Gamma(z) := \Theta_\Gamma(z/i)$ is a modular form of weight $n/2$ for $\mathrm{PSL}_2(\mathbf{Z})$.

This can be generalized in several directions. For instance, if we require only that $(\mathbf{x},\mathbf{y}) \in \mathbf{Q}$ for all $\mathbf{x},\mathbf{y} \in \Gamma$ then $\theta_\Gamma$ is still a modular form of weight $n/2$ for some congruence subgroup of $\mathrm{PSL}_2(\mathbf{Z})$. (See the Exercises.) We shall pursue a different generalization, applying Poisson to functions on $V$ more general than $\exp(-\pi t|\mathbf{x}|^2)$. We begin by introducing these functions.

**Harmonic polynomials.** Let $f : V \to \mathbf{C}$ be a function with $\int_V |f(\mathbf{x})|\, d\mathbf{x} < \infty$. Recall that we defined the Fourier transform $\hat{f} : V \to \mathbf{C}$ by

$$\hat{f}(\mathbf{y}) = \int_{\mathbf{x} \in V} e^{2\pi i (\mathbf{x}, \mathbf{y})} f(\mathbf{x})\, d\mathbf{x}.$$

We review some familiar properties of the Fourier transform. We shall apply these only to functions in the *Schwartz space* $\mathcal{S}$, consisting of infinitely differentiable functions on $V$ each of whose derivatives is $O((1 + |\mathbf{x}|)^{-N})$ for all $N$. We thus assume that $f \in \mathcal{S}$, in which case justification of all these properties is straightforward.

Let $x_1, \ldots, x_n$ be orthonormal coordinate functions on $V$, so $\mathbf{x} = (x_1, \ldots, x_n)$. Differentiating the integral for $\hat{f}$ with respect to $x_j$, we find that the the $j$-th partial derivative of $\hat{f}$ equals $2\pi i$ times the Fourier transform of $x_j f(\mathbf{x})$. Conversely, by integrating by parts with respect to $x_j$ we find that the Fourier transform of the $j$-th partial derivative $f_j$ equals $-2\pi i y_j \hat{f}$. This immediately yields the observation that if $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$ as well. The *Laplacian* $Lf$ of $f$ is defined by $Lf = \sum_{j=1}^n \partial^2 f / \partial x_j^2$, and does not depend on the orthonormal coordinate system. We see that the Fourier transform of $Lf$ equals $-4\pi^2 |\mathbf{y}|^2 \hat{f}$, and that the Fourier transform of $|\mathbf{x}|^2 f$ equals $-(4\pi^2)^{-1} L\hat{f}$.

We know that, for any $t \in \mathbf{C}$ with $\mathrm{Re}(t) > 0$, the function $f(\mathbf{x}) = \exp(-\pi t |\mathbf{x}|^2)$ has Fourier transform $\hat{f}(\mathbf{y}) = t^{-n/2} \exp(-\pi t^{-1} |\mathbf{y}|^2)$. By induction we can describe the Fourier transforms of functions of the form $P(\mathbf{x}) \exp(-\pi t |\mathbf{x}|^2)$, where $P$ is any polynomial in $n$ variables.

**Lemma.** *For any polynomial $P$ on $V$, the function $P(\mathbf{x}) \exp(-\pi t |\mathbf{x}|^2)$ on $V$ has Fourier transform $Q(\mathbf{y}) \exp(-\pi t^{-1} |\mathbf{y}|^2)$ for some polynomial $Q$ depending on $P$ and $t$. Moreover, if $P$ is of degree $d$ then so is $Q$, and the $d$-th homogeneous part of $Q$ is $(i/t)^d t^{-n/2}$ times the $d$-th homogeneous part of $P$.*

*Proof*: By linearity it is enough to prove this when $P$ is a monomial $\prod_{j=1}^n x_j^{d_j}$. In that case $P(\mathbf{x}) \exp(-\pi t |\mathbf{x}|^2)$ factors as $\prod_{j=1}^n x_j^{d_j} e^{-\pi t x_j^2}$, so its Fourier transform is the product of the Fourier transforms of the functions $x_j^{d_j} e^{-\pi t x_j^2}$. Let $f_d(y)$, then, be the Fourier transform of $x^d e^{-\pi t x^2}$. We know that $f_0(y) = t^{-1/2} e^{-\pi t^{-1} y^2}$ and $f_{d+1}(y) = (2\pi i)^{-1} f_d'(y)$. It follows inductively that $f_d(y) = Q_d(y) e^{-\pi t^{-1} y^2}$ for some polynomial $Q_d$ of degree $d$ and leading coefficient $t^{-d-\frac{1}{2}}$. Indeed $Q_0 = t^{-1/2}$ and $Q_{d+1} = i Q_d / t + Q_d' / 2\pi i$. The assertion of the Lemma thus holds for $P(\mathbf{x}) = \prod_{j=1}^n x_j^{d_j}$, when $Q(\mathbf{y}) = \prod_{j=1}^n Q_d(y_j)$, and therefore by linearity for all $P$. $\square$

Now suppose $P$ is homogeneous of degree $d$. In general $Q$ will not be homogeneous, except for $d = 0$ and $d = 1$. We shall show that the space of polynomials $P$ for which $Q$ is homogeneous, and thus equal to $(i/t)^d t^{-n/2}$, coincides with the space of *harmonic* polynomials of degree $d$, those whose Laplacian vanishes.

**Proposition.** *Let $P$ be a homogeneous polynomial of degree $d$ on $V$. Then $LP = 0$ if and only if the Fourier transform of the function $P(\mathbf{x})\exp(-\pi t|\mathbf{x}|^2)$ is $(i/t)^d t^{-n/2} P(\mathbf{y})\exp(-\pi t^{-1}|\mathbf{y}|^2)$.*

*Proof*: Recall that the gradient $\nabla f$ of a differentiable function $f$ on $V$ is defined by $\nabla f = \sum_{j=1}^n \mathbf{e}_j\, \partial f/\partial x_j$, and that a polynomial $P$ on $V$ is homogeneous of degree $d$ if and only if $\nabla P \cdot \mathbf{x} = d \cdot P$. Using the product formula $\nabla(\alpha\beta) = \alpha\nabla\beta + \beta\nabla\alpha$ we have

$$\nabla(P(\mathbf{x})e^{-\pi t|\mathbf{x}|^2}) = (\nabla P(\mathbf{x}) - 2\pi t\mathbf{x}P(\mathbf{x}))e^{-\pi t|\mathbf{x}|^2};$$

hence $P$ is homogeneous of degree $d$ if and only if

$$\nabla(P(\mathbf{x})e^{-\pi t|\mathbf{x}|^2}) \cdot \mathbf{x} = (d - 2\pi t|\mathbf{x}|^2)P(\mathbf{x})e^{-\pi t|\mathbf{x}|^2}. \tag{8}$$

We now apply the Fourier transform to both sides. For any $f \in \mathcal{S}$, the Fourier transform of $\nabla f(\mathbf{x})$ is $-2\pi i\mathbf{y}\hat{f}(\mathbf{y})$, so the Fourier transform of $\nabla f(\mathbf{x}) \cdot \mathbf{x}$ is

$$-\sum_{j=1}^n \partial(y_j f)/\partial y_j = -nf - \nabla f(\mathbf{y}) \cdot \mathbf{y}.$$

Thus the left-hand side of (8) has Fourier transform

$$-nQ(\mathbf{y})e^{-\pi t^{-1}|\mathbf{y}|^2} - \nabla(Q(\mathbf{y})e^{-\pi t^{-1}|\mathbf{y}|^2})\cdot\mathbf{y} = -(n+d-\frac{2\pi}{t}|\mathbf{y}|^2)Q(\mathbf{y})e^{-\pi t|\mathbf{y}|^2}, \tag{9}$$

where $Q(\mathbf{y})$ is the polynomial such that $Q(\mathbf{y})e^{-\pi t^{-1}|\mathbf{y}|^2}$ is the Fourier transform of $P(\mathbf{x})e^{-\pi t|\mathbf{x}|^2}$. The right-hand side has Fourier transform

$$(d + \frac{t}{2\pi}L)(Q(\mathbf{y})e^{-\pi t^{-1}|\mathbf{y}|^2},$$

which we evaluate using the product formula for the Laplacian:

$$L(\alpha\beta) = \alpha L\beta + \beta L\alpha + 2\nabla\alpha \cdot \nabla\beta,$$

obtaining

$$\left[(d + \frac{2\pi}{t}|\mathbf{y}|^2 - n)Q - 2\nabla Q \cdot \mathbf{y} + \frac{t}{2\pi}LQ\right] e^{-\pi t^{-1}|\mathbf{y}|^2}.$$

Comparing with (9), we find at last that

$$LQ(\mathbf{y}) = \frac{4\pi}{t}(\nabla Q(\mathbf{y}) \cdot \mathbf{y} - d \cdot Q(\mathbf{y})). \tag{10}$$

If $Q$ is a multiple of $P$ then it is homogeneous, so $\nabla Q(\mathbf{y}) \cdot \mathbf{y} = d \cdot Q(\mathbf{y})$. Hence $Q$, and thus also $P$, is harmonic. Conversely, suppose that $P$ is harmonic. Then $LP(\mathbf{x}) = 0 = (4\pi/t)\nabla P(\mathbf{x}) \cdot \mathbf{x} - d \cdot P(\mathbf{x})$. We multiply by $\exp(-\pi t|\mathbf{x}|^2)$ and apply the Fourier transform to both sides. Thus has the effect of running through our derivation of (10) in reverse, with the roles of $\mathbf{x}$ and $\mathbf{y}$ switched and with $t$ replaced with $t^{-1}$. We conclude that $\nabla Q(\mathbf{y}) \cdot \mathbf{y} = d \cdot Q(\mathbf{y})$, and thus that $Q$ is homogeneous of degree $d$. By our earlier Lemma, $Q$ thus equals $(i/t)^d t^{-n/2} P$. □

The harmonic polynomials of degree $d$ appear also as an irreducible representation of the orthogonal group $O(V)$; their restriction to the unit sphere of $V$ are the "spherical harmonics" of degree $d$, which constitute an eigenspace for the Laplacian on that sphere. They may also be defined as the space of homogeneous polynomials of degree $d$ whose restriction to the unit sphere is orthogonal to all the polynomials of degree $< d$.

If $P$ is a harmonic polynomial, we may apply the Poisson summation formula to $P(\mathbf{x}) \exp(-\pi t |\mathbf{x}|^2)$ to find the following generalization of Serre's Prop.16:

**Proposition.** *Let $\Gamma$ be a lattice in the $n$-dimensional real inner-product space $V$, with dual lattice $\Gamma'$ and covolume $v$. For any harmonic polynomial $P$ of degree $d$ on $V$, define the weighted theta function $\Theta_{\Gamma,P}$ by*

$$\Theta_{\Gamma,P}(t) := \sum_{\mathbf{x} \in \Gamma} P(\mathbf{x}) \exp(-\pi t |\mathbf{x}|^2) \quad (\mathrm{Re}(t) > 0).$$

*Then*

$$\Theta_{\Gamma,P}(t) = (i/t)^d t^{-n/2} v^{-1} \Theta_{\Gamma',P}(t^{-1})$$

*for all $t$ of positive real part.*

Note that this Proposition is nontrivial only for even $d$, because $\Theta_{\Gamma,P} = 0$ identically for $P$ of odd degree (why?). In particular we have:

**Corollary.** *If moreover $\Gamma$ is an even unimodular lattice then*

$$\theta_{\Gamma,P}(z) := \Theta_{\Gamma,P}(z/i) = \sum_{\mathbf{x} \in \Gamma} P(\mathbf{x}) q^{\frac{1}{2}|\mathbf{x}|^2}$$

*is a modular form of weight $(n/2) + d$ for $\mathrm{PSL}_2(\mathbf{Z})$.*

Again this generalizes to other lattices: if $\Gamma \subset V$ is a lattice such that $(\mathbf{x}, \mathbf{x}) \in \mathbf{Q}$ for all $x \in \Gamma$, and $\mathbf{x}_0$ is any vector in $\Gamma \otimes \mathbf{Q}$, then

$$\theta_{\Gamma,P,\mathbf{x}_0}(z) := \sum_{\mathbf{x} \in \Gamma + \mathbf{x}_0} P(\mathbf{x}) e^{\pi i (\mathbf{x},\mathbf{x}) z}$$

is a modular form of weight $(n/2)+d$ for some congruence subgroup of $\mathrm{PSL}_2(\mathbf{Z})$. Unless $2\mathbf{x}_0 \in \Gamma$, this form is usually nontrivial even for odd $d$. We have in effect already seen this for $d = n = 1$ when we introduced the modified theta functions $\vartheta_\chi(u)$ prove the functional equation for $L(s, \chi)$ when $\chi(-1) = -1$.

One useful application of this construction is the construction of "spherical $r$-designs". These are finite subsets $S$ of the unit sphere in $V$ that are very well distributed in the following sense: for any polynomial $P$ of degree at most $r$, the average of $P$ over the unit sphere equals $|S|^{-1} \sum_{\mathbf{x} \in S} P(\mathbf{x})$. It is known that this is equivalent to the condition that $\sum_{\mathbf{x} \in S} P(\mathbf{x}) = 0$ for all nonconstant harmonic polynomials $P$ of degree at most $r$. (See the Exercises.) For example, if $n = 2$ then the vertices of a regular $N$-gon constitute a spherical $(N-1)$-design.

**Proposition.** *Let $\Gamma$ is a self-dual unimodular lattice in an $n$-dimensional real inner product space. Assume that $\Gamma$ contains no nonzero vector $\mathbf{x}$ with $6|\mathbf{x}|^2 \leq (n/2) + r$. Then, for each $\rho > 0$ such that $\Gamma$ has vectors of length $\rho$, the set $\{\mathbf{x}/\rho : \mathbf{x} \in \Gamma, |\mathbf{x}| = \rho\}$ is a spherical $r$-design.*

*Proof*: Let $P$ be a nonconstant harmonic polynomial of degree $d \leq r$. We have seen that $\theta_{\Gamma,P}$ is a modular form of weight $(n/2) + d$. We claim that this form is identically zero. The Proposition will follow because the $q^{\rho^2/2}$ coefficient of $\theta_{\Gamma,P}$ is $\rho^d$ times the sum of $P(\mathbf{x}/\rho)$ over lattice vectors $\mathbf{x}$ of length $\rho$.

Since $d > 0$, the constant coefficient of $\theta_{\Gamma,P}$ vanishes. By hypothesis all the other terms $P(\mathbf{x})q^{(\mathbf{x},\mathbf{x})/2}$ in the sum for $\theta_{\Gamma,P}$ have exponent greater than $((n/2)+r)/6$. But a cusp form of weight $w$ that vanishes to order $> w/6$ at the cusp must be identically zero. $\square$

It is known that $r$ cannot exceed 11 for any $\Gamma$, and that if $r = 11$ then $24|n$ and $n$ is bounded above. Examples are known only for $n = 24$ (the celebrated Leech lattice) and $n = 48$. (See for instance [CS 1993].) In addition to providing good spherical codes for numerical integration, this Proposition and its generalizations to other kinds of lattices has been used to investigate the existence and uniqueness of certain lattices, notably in the four-page proof [Conway 1969] that the Leech lattice is the unique even unimodular lattice in $\mathbf{R}^{24}$ with no vectors of length $\sqrt{2}$.

**Exercises**

Concerning modular groups and their rings of modular forms:

1. i) Prove Serre's assertion (bottom of p.111) that $S$ and $T^2$ generate an index-3 subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ by identifying this subgroup with $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbf{Z}) : 2|b\}$.
ii) Find similar pairs of generators for $\Gamma_0(N)$ $(N = 2, 3, 4)$, where

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{Z}) : N|c \right\}.$$

In particular, show that $\Gamma_0(4)$ is a free group on two generators.
iii) What are the groups generated by $T$ and $S_d : z \leftrightarrow -1/dz$ for $d = 2, 3, 4$?

2. For each of the groups in the previous Exercise, determine the ring of modular forms and the ideal of cusp forms in that ring.

[There are only finitely many congruence groups that can be treated in this way; can you find any others? See [Takeuchi 1977] for more information. The group generated by $T$ and $S_d$ arises naturally in the analysis of theta functions of lattices $\Gamma$ with $\Gamma' \cong d^{-1/2}\Gamma$, sometimes called "isodual lattices".]

Some more identities and results involving one-dimensional theta functions:

3. For the purposes of this problem,[2] define

$$\theta_0(z) := \sum_{m=-\infty}^{\infty} q^{m^2/2}, \qquad \theta_1(z) := \sum_{m=-\infty}^{\infty} (-1)^m q^{m^2/2}.$$

i) How do $\theta_0$ and $\theta_1$ transform under $S$ and $T$?
ii) Prove that

$$2\theta_0^2(2z) = \theta_0^2(z) + \theta_1^2(z), \qquad \theta_1^2(2z) = \theta_0(z)\theta_1(z).$$

Thus $(x_n, y_n) = c(\theta_0^2(2^n z), \theta_1^2(2^n z))$ is a solution of the AGM (Arithmetic-Geometric Mean) recurrence $(x_{n+1}, y_{n+1}) = (\frac{1}{2}(x_n + y_n), (x_n y_n)^{1/2})$.
[This recursion converges quadratically to $(c, c)$; that is, $|(x_{n+1} - c, y_{n+1} - c)| \ll |(x_n - c, y_n - c)|^2$. The AGM map $(x, y) \mapsto ((x + y)/2, \sqrt{xy}$ is closely related to "Landen's transformation" of elliptic integrals, which in modern terminology realizes a "2-isogeny" between elliptic curves. See [BB 1987] for some theoretical and computational uses of this identity.]
iii) What is the weight-12 modular form $\theta_0^4\theta_1^4(\theta_0^4 - \theta_1^4)^4$?
[See the expository article [Elkies 2000] for one interpretation of this formula; the forms $\theta_0, \theta_1$ that appear there are related with ours via the result of part (i) of this Exercise.]

4. Prove that for any $s \in \mathbf{Q}$ the shifted theta function $\sum_{n\in\mathbf{Z}} \exp(pii(n + s)^2 z)$ is a modular form for some congruence subgroup of $\mathrm{PSL}_2(\mathbf{Z})$. Use this to show that if $\Gamma$ is any lattice such that $(\mathbf{x}, \mathbf{y}) \in \mathbf{Q}$ for all $\mathbf{x}, \mathbf{y} \in \Gamma$ then $\theta_\Gamma$ is a modular form of (possibly half-integral) weight $\dim(V)/2$ for some congruence subgroup of $\mathrm{PSL}_2(\mathbf{Z})$. [Hint: Find an orthogonal basis of $V$ consisting of lattice vectors.]

Concerning product formulas:

5. i) Evaluate the integrals (2) and (3) directly and verify that the difference between them is $2\pi i/z$.
ii) Instead of $\sum_{n=-N}^{N} \sum'_m - \sum_{m=-N}^{N} \sum'_n$, we could have taken the limit of $\sum_{n=-aN}^{bN} \sum'_m - \sum_{m=-cN}^{dn} \sum'_n$ for any positive $a, b, c, d$; what would happen then?

6. Prove that

$$\sum_{n=1}^{\infty} n\chi_4(n) q^{n^2} = \eta(8z)^3 \left[ = q \prod_{n=1}^{\infty} (1 - q^{8n})^3 \right]$$

for $|q| < 1$. Note that the sum $\sum_{n=1}^{\infty} n\chi_4(n) q^{n^2} = q - 3q^9 + 5q^{25} - 7q^{49} + - \cdots$ is closely related to the function that we called $\vartheta_{\chi_4}$ in the proof of the functional equation for $L(s, \chi_4)$. [In general, for any Dirichlet character $\chi \bmod q$, the function $\vartheta_\chi(z/qi)$ is a modular form of weight 3/2 for some congruence subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ depending on $q$.] Can you guess and prove product formulas for $\vartheta_{\chi_3}$ and $\theta_{\chi_8}$, and for that matter for the untwisted theta function $\sum_{n=-\infty}^{\infty} q^{n^2}$?

[2] Traditionally $\theta_i$ $(i = 0, 1, 2, 3)$ is used for a different family of theta series.

7. Prove that

$$\sum\sum_{m,m'\in\mathbf{Z}} mm'(m+m')\chi_3(m)q^{m^2+mm'+m'^2} = 12\,\eta(9z)^8 \left[ = 12q^3\prod_{n=1}^{\infty}(1-q^{9n})^8\right].$$

Concerning harmonic polynomials etc.:

8. Complete the proof that if $P(\mathbf{x})$ is harmonic then $P(\mathbf{x})\exp(-\pi t|\mathbf{x}|^2)$ has Fourier transform $(i/t)^d t^{-n/2}P(\mathbf{y})\exp(-\pi t^{-1}|\mathbf{y}|^2)$.

9. Prove that the Laplacian is a surjective linear operator on the vector space $\mathbf{C}[V]$ of polynomials on $V$. Conclude that the space of harmonic polynomials of degree $d$ on an $n$-dimensional real inner-product space is the difference $\binom{n+d-1}{d} - \binom{n+d-3}{d-2}$ between the dimensions of homogeneous polynomials of degrees $d$ and $d-2$.

10. Show that the only homogeneous polynomial $P(\mathbf{x})$ such that $|\mathbf{x}|^2 P(\mathbf{x})$ is harmonic is the zero polynomial. Deduce that every homogeneous polynomial $P(\mathbf{x})$ of degree $d$ can be written uniquely as $\sum_{j=0}^{\lfloor d/2\rfloor} |\mathbf{x}|^{2j} P_j(\mathbf{x})$ with the $P_j$ being harmonic polynomials of degree $d-2j$. Conclude that the vector space of polynomial functions of degree at most $d$ on the unit sphere in $V$ is the direct sum of the spaces of spherical harmonics of degree $d'$ with $d' \leq d$, and thus that a finite set $S$ of unit vectors is an $r$-design if and only if $\sum_{\mathbf{x}\in S} P(\mathbf{x}) = 0$ for all nonconstant harmonic polynomials $P$ of degree at most $r$.

11. Prove that, given the dimension of $V$, there exists for each $d = 0, 1, 2, \ldots$ a unique monic $P_d \in \mathbf{R}[x]$ of degree $d$ such that $|\mathbf{x}|^d P_d(x_1/|\mathbf{x}|)$ is a harmonic polynomial. [This $P_d$ is proportional to a Gegenbauer orthogonal polynomial, and the spherical harmonic $P_d(x_1)$ on the unit sphere of $V$ is called a "zonal spherical harmonic". These have many applications in mathematics, including those of the next Exercise.]

12. i) Let $\Gamma$ be an even unimodular lattice in $\mathbf{R}^{32}$ with no vectors of length $\sqrt{2}$. Use the modularity of $\theta_\Gamma$ to prove that $\Gamma$ has 146880 vectors of the minimal nonzero length 2. Use theta functions weighted by the polynomials of the previous Exercise to prove that if $v_0$ is one of those vectors then $N_k$ vectors $v$ of length 2 such that $(v, v_0) = k$, where $N_0 = 80910$, $N_{\pm 1} = 31744$, $N_{\pm 2} = 240$, $N_{\pm 4} = 1$, and $N_k = 0$ for all $k \notin \{0, \pm 1, \pm 2, \pm 4\}$. Obtain similar results for the distribution of $(v, v_1)$ where $v_1$ is a fixed lattice vector of length $\sqrt{6}$ and $v \in \Gamma$ has length 2, or vice versa.
ii) Can you obtain similar conditions on the 39600 minimal vectors of an even unimodular lattice in $\mathbf{R}^{40}$ with no vectors of length $\sqrt{2}$?
iii) If $\Gamma$ is an even unimodular lattice in $\mathbf{R}^{48}$ whose shortest nonzero vectors have length $\sqrt{6}$, or in $\mathbf{R}^{72}$ whose shortest nonzero vectors have length $\sqrt{8}$, how much combinatorial data can you obtain about the configurations of short vectors in $\Gamma$?

[It is known that there are several such "extremal" lattices $\Gamma$ in dimension 32, and several in dimension 48. Their classification is not yet complete, but all must satisfy these combinatorial constraints. There are literally millions of inequivalent such lattices in dimension 40. It is not yet known whether there exists an extremal lattice in dimension 72. See again [CS 1993].]

13. Prove that a set of $N$ unit vectors in a 2-dimensional real inner-product space is a spherical $(N-1)$-design[3] if and only if it comprises the vertices of a regular $N$-gon.

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[3] One might say "circular $(N-1)$-design" in this setting. . .