

Math 259: Introduction to Analytic Number Theory

How small can $|\text{disc}(K)|$ be for a number field K of degree $n = r_1 + 2r_2$?

Let K be a number field of degree $n = r_1 + 2r_2$, where as usual r_1 and r_2 are respectively the numbers of real embeddings and conjugate complex embeddings of K . Let O_K be the ring of algebraic integers of K , and $D_K = \text{disc}(K/\mathbf{Q})$ the discriminant. Minkowski proved that every ideal class of K contains some ideal $J \subseteq O_K$ of norm at most $(n!/n^n)(4/\pi)^{r_2}|D_K|^{1/2}$. (See for instance [Marcus 1977].) In particular, the principal ideal class contains such a J (which might as well be taken to be O_K itself), and since the norm of J is at least 1 we recover the *Minkowski bound*

$$|D_K| \geq \left(\frac{\pi}{4}\right)^{2r_2} \left(\frac{n^n}{n!}\right)^2. \quad (1)$$

In particular, it readily follows that $|D_K| > 1$ once $n > 1$ (that is, except for $K = \mathbf{Q}$); that is, \mathbf{Q} has no nontrivial unramified extension. This is a key ingredient of the Kronecker-Weber theorem, which asserts that any finite extension of \mathbf{Q} with abelian Galois group is contained in a cyclotomic extension $\mathbf{Q}(e^{2\pi i/n})$.

Asymptotically as $n \rightarrow \infty$, Minkowski's bound is

$$\log |D_K| \geq (2 - o(1))n - 2 \log(\pi/4)r_2. \quad (2)$$

That is, we have the lower bound $(\pi/4)^{2r_2/n}e^2 - o(1)$ on the “root-discriminant” $|D_K|^{1/n}$. (Note for future reference the numerical values: $(\pi/4)^{2r_2/n}e^2$ is approximately $(7.389)^{r_1/n}(5.803)^{2r_2/n}$.) It is known that the root-discriminant is invariant under unramified extensions; for instance (1) also implies that some other number fields — such as the quadratic fields $\mathbf{Q}(e^{2\pi i/3})$, $\mathbf{Q}(i)$, $\mathbf{Q}(\sqrt{5})$ whose discriminants $-3, -4, 5$ have the smallest absolute values — have no nontrivial unramified extension. Subsequent work extended Minkowski's “geometry of numbers” to show $\log |D_K|$ is bounded below by larger linear combinations of r_1, r_2 .

In the other direction, Golod and Šafarevič proved that quadratic number fields K_0 whose discriminants have many prime factors have an infinite “class field tower”, and thus unramified extensions K with $[K : K_0] \rightarrow \infty$. Such K all have root-discriminant $|D_{K_0}|^{1/2}$. There is thus an upper limit to improvements on the constants in (2). One survey of such constructions and the resulting upper limits is [Schoof 1986].

Much less is known here than for the analogous question on curves C of high genus with many points over a fixed finite field k . (See the Remarks below.) The best lower bounds for all but the smallest few n are now obtained by a method independent of Minkowski's approach, and similar to the techniques that yield upper bounds on $\#C(k)$. The method, attributed to Stark [1974, 1975] by Odlyzko [1991], uses the Euler and Hadamard products for the zeta function ζ_K to transform the functional equation for ζ_K into a formula for $\log |D_K|$ in

terms of r_1, r_2 , and the nontrivial zeros of ζ_K . In a series of papers starting from [Odlyzko 1975], the bounds were progressively improved until reaching their present form:

Theorem. *Let K be a number field of degree $n = r_1 + 2r_2$. Then*

$$\log |D_K| > (\log 4\pi + \gamma - o(1))n + r_1 \quad (3)$$

as $n \rightarrow \infty$, where $\gamma = -\Gamma'(1) = .577\dots$ is Euler's constant. If moreover ζ_K satisfies the Generalized Riemann Hypothesis then

$$\log |D_K| > (\log 8\pi + \gamma - o(1))n + (\pi/2)r_1 \quad (4)$$

as $n \rightarrow \infty$.

Numerically, the root-discriminant of K is asymptotically bounded below by $(60.8)^{r_1/n}(22.38)^{2r_2/n}$, and by $(215.3)^{r_1/n}(44.7)^{2r_2/n}$ under the GRH. For many applications one needs also explicit estimates on the $o(1)$ terms for specific values of (r_1, r_2) . Odlyzko carried out extensive numerical computations to obtain good lower bounds for many (r_1, r_2) . See [Odlyzko 1991] for a survey of the methods used and some of the applications, which include the theorem that each of the nine imaginary quadratic fields of class number 1 has no nontrivial unramified extensions. (NB the last of these fields has root-discriminant $\sqrt{163} < 13$.)

We present only a simple proof of the asymptotic estimate under GRH, making no attempt to optimize the $o(1)$ error. The same approach yields the unconditional bound (3); see the Exercises.

We begin by obtaining Artin's formula for $|D_K|$:

Proposition. *For all real $s > 1$ we have*

$$\begin{aligned} \log |D_K| = & r_1 \left(\log \pi - \frac{\Gamma'}{\Gamma}(s/2) \right) + 2r_2 \left(\log 2\pi - \frac{\Gamma'}{\Gamma}(s) \right) \\ & - \frac{2}{s-1} + \frac{2}{s} - \frac{\zeta'_K}{\zeta_K}(s) + 2 \sum_{\rho} \operatorname{Re} \frac{1}{s-\rho}, \end{aligned} \quad (5)$$

where ρ runs over the nontrivial zeros of $\zeta_K(s)$ counted with multiplicity.

Proof: Recall that the functional equation for ζ_K may be written in the form

$$\xi_K(s) := \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} (4^{-r_2} \pi^{-n} |D_K|)^{s/2} \zeta_K(s) = \xi_K(1-s), \quad (6)$$

and that $(s^2 - s)\xi_K(s)$ is an entire function of s of order 1. Translation by $1/2$ yields the entire function $(s^2 - \frac{1}{4})\xi_K(s + \frac{1}{2})$ symmetric under the involution $s \mapsto -s$. The logarithmic derivative of the Hadamard product for this function yields the partial-fraction decomposition

$$\frac{\xi'_K(s)}{\xi_K(s)} = B - \frac{1}{s} - \frac{1}{1-s} + \frac{m}{s - \frac{1}{2}} + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho - \frac{1}{2}} \right). \quad (7)$$

Here m is the multiplicity of the zero, if any, of $\zeta_K(s)$ at $s = 1/2$; and ρ runs over the nontrivial zeros of $\zeta_K(s)$ counted with multiplicity, excluding $1/2$. Since (7)

is an odd function of $s - 1/2$, the constant B must vanish. We can now take the logarithmic derivative of (6) and solve for $|D_K|$. Averaging the ρ and $1 - \rho$ terms, we may replace the summand $(s - \rho)^{-1} + (\rho - 1/2)^{-1}$ by

$$\frac{1}{2} \left(\frac{1}{s - \rho} + \frac{1}{s - (1 - \rho)} \right).$$

Having thus eliminated $(\rho - 1/2)^{-1}$, we recover (6) as follows. For real s , $\zeta'_K(s)/\zeta_K(s) \in \mathbf{R}$, so we may replace $((s - \rho)^{-1} + (s - 1 + \rho)^{-1})/2$ by its real part. Since each ρ has real part in $(0, 1)$, we have $\operatorname{Re}(1/(s - \rho)) \ll \operatorname{Im}(\rho)^{-2}$ as $|\rho| \rightarrow \infty$, so $\sum_{\rho} \operatorname{Re}(1/(s - \rho))$ converges absolutely. We may therefore rearrange the sum, including the contribution of a possible zero at $1/2$, to obtain (6). \square

We now proceed to the *Proof* of (4). By the Euler product, $-\zeta'_K/\zeta_K$ and its derivatives of even order with respect to s are positive for $s > 1$, and the derivatives of odd order negative; thus by differentiating (6) m times ($m = 0, 1, 2, \dots$) we find

$$\begin{aligned} \delta_m \log |D_K| > (-1)^m \left[r_1 \frac{d^m}{ds^m} \left(\log \pi - \frac{\Gamma'}{\Gamma}(s/2) \right) + 2r_2 \frac{d^m}{ds^m} \left(\log 2\pi - \frac{\Gamma'}{\Gamma}(s) \right) \right] \\ + m! \left(2 \sum_{\rho} \operatorname{Re} \frac{1}{(s - \rho)^{m+1}} - \frac{2}{(s - 1)^{m+1}} + \frac{2}{s^{m+1}} \right). \end{aligned} \quad (8)$$

(Here δ_m is a form of Kronecker's delta, which equals 1 for $m = 0$ and zero otherwise.)

Our idea is now that for fixed $s > 1$ and large n the term in $(s - 1)^{-(m+1)}$ is negligible, and so by dividing the rest of (8) by $2^m m!$ and summing over m we obtain (6) with s replaced by $s - 1/2$ (Taylor expansion about s). Since $\operatorname{Re}(1/(s - \frac{1}{2} - \rho))$ is still positive, we then find by bringing s arbitrarily close to 1 that

$$\log |D_K| > r_1 \left(\log \pi - \frac{\Gamma'}{\Gamma}(1/4) \right) + 2r_2 \left(\log 2\pi - \frac{\Gamma'}{\Gamma}(1/2) \right) - o(n),$$

and thus obtain our Theorem from the known special values

$$\frac{\Gamma'}{\Gamma}(1/2) = -\log 4 - \gamma, \quad \frac{\Gamma'}{\Gamma}(1/4) = -\log 8 - \pi/2 - \gamma. \quad (9)$$

To make this rigorous, we argue as follows. For any small $\epsilon > 0$, take $s_0 = 1 + \epsilon$, and pick an integer M so large that

- (i) the values at $s = s_0 - 1/2$ of the M -th partial sums of the Taylor expansions of $(\Gamma'/\Gamma)(s)$ and $(\Gamma'/\Gamma)(s/2)$ about $s = s_0$ are within ϵ of $(\Gamma'/\Gamma)(s_0 - 1/2)$ and $(\Gamma'/\Gamma)(s_0/2 - 1/4)$ respectively;
- (ii) the value at $s = s_0 - 1/2$ of the M -th partial sum of the Taylor expansion of $\operatorname{Re}(1/(s - \frac{1}{2} - \rho))$ about $s = s_0$ is positive for all complex numbers ρ of real part $1/2$.

Condition (i) holds for large enough M because $(\Gamma'/\Gamma)(s)$ and $(\Gamma'/\Gamma)(s/2)$ are both analytic functions of s in a circle of radius $1 > 1/2$ about s_0 . To verify that (ii) also holds as $M \rightarrow \infty$, let¹ $\rho = 1/2 + it$, and note that $\operatorname{Re}(1/(s - \frac{1}{2} - \rho)) = \operatorname{Re}(1/(s - it)) = \epsilon/(\epsilon^2 + \operatorname{Im}(\rho)^2)$, and the value of the M -th partial sum of the Taylor expansion differs from this by

$$\operatorname{Re} \frac{1}{[1 + 2(\epsilon - it)]^M (\epsilon + it)} \ll (1 + \epsilon^2 + t^2)^{-M/2}.$$

The positive $\epsilon/(\epsilon^2 + t^2)$ clearly dominates the error $(1 + \epsilon^2 + t^2)^{-M/2}$ uniformly in t once M is sufficiently large.

Now divide (8) by $2^m m!$, sum from $m = 0$ to $M - 1$, and set $s = s_0$ to obtain

$$\log |D_K| > r_1 \left(\log \pi - \frac{\Gamma'}{\Gamma}(s_0/2 - 1/4) - \epsilon \right) + 2r_2 \left(\log 2\pi - \frac{\Gamma'}{\Gamma}(s_0 - 1/2) - \epsilon \right) + O(1);$$

since ϵ was arbitrarily small and s_0 arbitrarily close to 1, we are done. $\square\square$

Remarks

Besides the problem of evaluating limits such as $\liminf_{n \rightarrow \infty} \log |D_K|/n$, many other natural questions remain wide open in this context where analogous questions for high-genus curves with many rational points over a finite field have been settled for some time. We list several of these open questions:

- It is not known how to construct class field towers explicitly. Can one construct an explicit infinite sequence of number fields K with bounded root-discriminant?
- When a class field tower over K_0 can be proved infinite, the resulting unramified extensions K have $[K : K_0]$ limited to a very sparse set of positive integers, namely those whose prime factors are contained in a given finite set S . Does there exist $\theta > 0$ an infinite sequence of number fields K with bounded root-discriminant whose degrees cover at least x^θ of the integers $n < x$ as $x \rightarrow \infty$?
- More ambitiously: Can there be such a sequence that covers *every* n ? Equivalently, is $\limsup_{n \rightarrow \infty} \log |D_K|/n$ finite?
- In another direction: in a class field tower over a fixed number field, the ratios r_1/n are limited to a small subset of $[0, 1] \cap \mathbf{Q}$. Does there exist a finite R such that the number fields K with $|D_K| < R^n$ have ratios r_1/n that form a dense subset of $[0, 1]$, or even of an interval of positive length in $[0, 1]$?
- The ratio r_1/n can be regarded as a measure of the behavior of the “archimedean place” of \mathbf{Q} in K . Similar questions can be posed concerning the splitting or ramification of a given set of “nonarchimedean places” (rational primes) in K .

¹The customary $\rho = 1/2 + i\gamma$ may lead to confusion in the presence of Euler’s constant γ .

Another notable application of the method of Odlyzko et al. is Mestre's lower bound on the conductor of an elliptic curve E/\mathbf{Q} of given rank, assuming GRH as well as the conjecture of Birch and Swinnerton-Dyer for the L -function $L(E, s)$ of the curve. Similar bounds have been obtained for even more complicated L -functions.

Exercises

1. Fill in the missing steps in our proof of (4) by checking the derivation of the formula (6) or $\log |D_K|$ and proving the formulas (9) for the logarithmic derivative of $\Gamma(s)$ at $s = 1/2$ and $s = 1/4$.
2. Show that the Odlyzko bound (4) still holds under the weakened hypothesis that all zeros of $\zeta_K(s)$ are either real or on the critical line $\sigma = 1/2$. (This hypothesis allows also for nontrivial zeros on $(0, 1)$.) Can you find a yet weaker hypothesis on the zeros under which (4) remains true?
3. Use the same methods to prove the unconditional lower bound (3).
4. Suppose that the rational prime 2 splits completely in K (whence the Euler product for $\zeta_K(s)$ contains the factor $(1 - 2^{-s})^{-n}$). Obtain lower bounds on $|D_K|$, both unconditionally and under GRH, that improve on (3,4). Generalize.

References

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