Math 2^8: The Theory of Error-Correcting Codes

Stationary-phase estimates for some power-series coefficients

We have run across several instances of the following problem: given power series

$$G = \sum_{k \geq -p} g_k q^k, \quad A = \sum_{k \geq k_0} a_k q^k$$

(with $p > 0$, this being the order of the pole of G at $q = 0$), how does the constant coefficient C_m of AG^m behave as $m \to \infty$? (If we're interested in some other coefficient q^b with b fixed we can recover it from the constant coefficient of $(q^{-b}A)G^m$, so the $b = 0$ question is actually no less general.)

For example, we found a formula for the kissing number of an extremal Type II code of length $24m$ in terms of C_m for $A = 3q(14+2q)(1+14q+q^2)^2$ and $G = \bigl(q(1-q)^4\bigr)^{-1}$, and will need the asymptotics of this kissing number and of the next coefficient of the extremal weight enumerator. More elementary examples, also with $p = 1$, include $(A, G) = (1, q^{-1} + g_1 q)$, when $C_m = g_1^{m/2}\binom{m}{m/2}$ or 0 according as m is even or odd (and likewise $(A, G) = q^{-1} + g_2 q^2$, giving $g_1^{m/3}\binom{m}{m/3}$ if $3|m$ and 0 otherwise), or $(A, G) = (q^i, G = q^{-1} + 1 + q^{-1})$, when C_m is the "king number" counting m-move paths of a chess king from $(0, 0)$ to (m, i).

So far we've used only formal manipulations of power series, but for asymptotic estimates we shall need our series to actually converge in some neighborhood of $q = 0$ (as indeed they do in all our examples). In each case that we work with, all the coefficients a_k and g_k are nonnegative. In that case the same is true for the coefficients of AG^m, so we obtain an upper bound on C_m by just evaluating at any $q_0 > 0$ in the circle of convergence: if the q^k coefficient of AG^m is $C_m^{(k)}$ then

$$C_m = C_m^{(0)} < \sum_{k=k_0-mp}^{\infty} C_m^{(k)} q_0^k = A(q_0)G(q_0)^m,$$

and as $m \to \infty$ the tightest (i.e. smallest) bound is obtained by choosing for q_0 the value that minimizes $G(q_0)$ (which happens at a unique positive q_0 as long as $a_k > 0$ for some $k > 0$ [NB we implicitly assume $a_{-p} > 0$], because G is convex upwards and $G(q) \to \infty$ as $q \to 0$ and also as q goes to the [possibly infinite] radius of convergence). We must thus assume that the power series defining $A(q)$ converges at this q_0. We shall see that, under a very mild further hypothesis, the elementary estimate $C_m < A(q_0)G(q_0)^m$ is within a factor of order $\sqrt{m}$ of the correct asymptotic:

$$(*) \qquad C_m \sim \frac{G(q_0)^m}{\sqrt{m}} \left(\alpha_0 + \frac{\alpha_1}{m} + \frac{\alpha_2}{m^2} + \cdots \right),$$

where

$$\alpha_0 = \sqrt{\frac{G(q_0)}{2\pi G''(q_0)}}\, A(q_0).$$

[Note that (*) starts "$C_m \sim \cdots$", not "$C_m = \cdots$"; it's an *asymptotic series*, which need not converge to C_m, or indeed to anything: as with Taylor series, an asymptotic series "$C_m \sim F(m) \sum_{j=0}^{\infty} \alpha_j / m^j$" means that for each j_0 the truncated sum $F(m) \sum_{j=0}^{j_0-1} \alpha_j / m^j$ is within $O(F_m / m^{j_0})$ of C_m as $m \to \infty$.] The "very mild hypothesis" is that the coefficients g_k not be supported on an arithmetic progression $k \equiv -p \bmod d$ for any $d > 1$. Note that this hypothesis fails for our examples with $G = q^{-1} + g_1 q$ and $G = q^{-1} + g_2 q^2$ (where $d = 2$ and $d = 3$) which led to C_m depending on $m \bmod d$ (and even when $C_m \neq 0$ our asymptotic formula is missing a factor of d). But it is easy to reduce to the case $d = 1$ by a change of variable, replacing $G(q)$ by $G(q^{1/d})^d$.

To prove (*) we begin by writing the constant coefficient as the average of AG^m over a circle:

$$C_m = \int_{-1/2}^{1/2} A(re^{2\pi i\theta})G(re^{2\pi i\theta})^m d\theta$$

for any $r > 0$ smaller than the radius of convergence of the power series $A(q)$ and $G(q)$. [This is equivalent to the usual formula using a contour integral around the circle $|q| = d$.] We'll fix r and let $m \to \infty$. The integrand's absolute value is maximized at $\theta = 0$, and thanks to our additional hypothesis on the a_k this maximum is unique on $\mathbf{R}/\mathbf{Z}$. But for most choices of r we see that increasing m makes the integrand oscillate ever faster at $\theta = 0$, causing massive cancellations that mask the actual asymptotic growth. The exception is the one value $r = q_0$ where G, and thus its phase $\operatorname{Im}\log G$, has a critical point (hence the name "stationary phase" for this asymptotic technique). We then have

$$G(re^{2\pi i\theta})^m = G(q_0) + 2\pi^2 G''(q_0)\theta^2 + O(\theta^3)$$

for small θ. So we expect that for large m and small θ the integrand behaves like $A(q_0)G(q_0)^m e^{-cm\theta^2}$ where $c = 2\pi^2 G''(q_0)/G(q_0)$, from which the formula $C_m \sim G(q_0)^m \alpha_0/\sqrt{m}$ will follow via the Gaussian integral $\int_{-\infty}^{\infty} e^{-cm\theta^2} d\theta = \sqrt{\pi/cm}$.

To prove this, and obtain further terms of the asymptotic expansion, apply a change of variables to $z = z(\theta)$ such that $G(q_0 e^{2\pi i\theta})$ is actually equal $G(q_0)e^{-z^2/2}$ in a neighborhood $|\theta| < \delta$ of $\theta = 0$. We then write

$$A(re^{2\pi i\theta})G(re^{2\pi i\theta})^m d\theta = \tilde{A}(z)e^{-mz^2/2}dz$$

in that neighborhood, and expand $\tilde{A}(z)$ in a Taylor series with constant term

$$\tilde{A}(0) = A(q_0)/z'(\theta) = 2\pi(G''(q_0)/G(q_0))^{1/2}.$$

For large m we may then estimate $\int_{z(-\delta)}^{z(\delta)} \tilde{A}(z)e^{-mz^2/2}dz$ by the integral of the same integrand form $z = -\infty$ to $+\infty$, and get our asymptotic expansion using the definite integral

$$\int_{-\infty}^{\infty} z^{2k}e^{-mz^2/2} = \sqrt{2\pi}\frac{(2k-1)!!}{m^k},$$

where "$(2k-1)!!$" is $1 \cdot 3 \cdot 5 \cdots (2k-1) = (2k)!/2^k k!$ (and of course the odd terms $z^{2k-1}e^{-mz^2/2}$ integrate to zero).

Exercises

1. Fill in our sketch of the proof of (*). [Warning: at $\theta = \pm\delta$ our new variable z will take complex variables, so the integral $\int_{z(-\delta)}^{z(\delta)}$ must be interpreted as a contour integral to be approximated by $\int_{z=-\infty}^{z=\infty}$.]

2. Adapt this technique to obtain the beginning of the refinement of Stirling's approximation for

$$n! = \Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx$$

to an asymptotic formula:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right)$$

as $n \to \infty$ (approximate the integrand by a Gaussian near its maximum at $x = n$, which is a stationary point, even though no phase is involved here).