Math 2^8: The Theory of Error-Correcting Codes

An upper bound on Euclidean sphere packing

We showed that any $n+1$ unit vectors v_i in a Euclidean space include two that satisfy $\langle v_i, v_j \rangle > -1/n$, and thus are at distance at most $(2+\frac{2}{n})^{1/2}$. Thus if we have $n+1$ vectors in the unit *ball* $|v| \leq 1$ then it is still true that two of them are at distance at most $(2+\frac{2}{n})^{1/2}$: if some $v_i = 0$ then it's clear, and otherwise apply the previous result to $u_i := v_i/|v_i|$ to find some v_i, v_j with $\langle v_i, v_j \rangle > -|v_i||v_j|/n$, from which the desired inequality follows.

Now if we have a collection of disjoint unit spheres in $\mathbf{R}^n$ then no open ball of radius $2/(2+\frac{2}{n})^{1/2}$ can contain more than n of their centers. Averaging over the center of the ball, we find that the center density is at most n divided by its volume, which is $n \cdot ((n+1)/2n)^{-n/2}$ divided by the volume of a unit sphere. Therefore the packing density is at most $n \cdot ((n+1)/2n)^{-n/2} \ll n2^{-n/2}$ as claimed.