Math 2^8: The Theory of Error-Correcting Codes

Problem Set #3:
Synthemes, Totals, and Π_5

1. These exercises outline a proof of the uniqueness of the projective plane of order 5 and the identification of its automorphism group with $\mathrm{PGL}_3(\mathbf{F}_5)$; fill in the details. Recall that an *oval* O in such a plane Π_5 is a set of 6 points with no three collinear. Begin by showing that the number of ovals in Π_5 is $31 \cdot 30 \cdot 25 \cdot 16 \cdot 6 \cdot 1/6! = 3100$. Fix one oval O. Of the $5^2+5+1=31$ lines of Π_5, six are tangent, $\binom{6}{2} = 15$ are "secant" (contain two points of O), and the remaining 10 "passant" (disjoint from O).

2. Of the $31 - 6 = 25$ points of $\Pi_5 - O$ let n_i be the number lying on i tangents (necessarily $2|i$). As before, compute that $\sum_{i>0} in_i = 6 \cdot 5 = 30$ and $\sum_{i>0}\binom{i}{2}n_i = \binom{6}{2} = 15$ to conclude that $n_2 = 15$ and $n_i = 0$ for all $i > 2$. Thus each pair of tangents meets in a different point; we call such a points a "tangential point" or T point for short. Each of the remaining $n_0 = 25 - 15 = 10$ points of Π_5 lies on no tangent, so is the intersection of three secants to O, determining a syntheme. We call such a point a "synthematic point" or S point, and let $\mathcal{S}$ be the set of ten synthemes determined by the S points.

3. Any secant meets the tangents to its endpoints on O, and the remaining $6-2 = 4$ tangents off O. Thus it meets them in pairs at $4/2$ or two T points. Therefore the remaining $6-2-2 = 2$ points on the secant are S points. It follows that each pair occurs twice in the $\mathcal{S}$ synthemes, whence $\mathcal{S}$ is the complement of a total.

4. Of the 6 lines through each T point, 2 are tangent and 2 secant to O, so the remaining 2 are passant. Thus the number of pairs P, l with P a T point and l a passant line through P is $15 \cdot 2 = 30$. But no passant line may go through more than three T points ($3 = 6/2$): any more than that include two that lie on a common tangent. Thus each of the 10 passant lines contains exactly three T points (and thus three S points since $6 - 3 = 3$).

5. Since T points correspond to pairs in O, each passant line determines via its T points three disjoint pairs, i.e. a syntheme. Let $\mathcal{S}'$ be the set of ten synthemes thus arising. We'll show $\mathcal{S}' = \mathcal{S}$. Indeed if $(AB)(CD)(EF)$ is a syntheme *not* in $\mathcal{S}$ than the T point at the intersection of the A, B tangents lies on the CD, EF secants; likewise for C, D. Thus the line joining these two T points is the EF secant, which does not contain the T point E, F, whence

$(AB)(CD)(EF) \notin \mathcal{S}'$. Thus $\mathcal{S}' \subseteq \mathcal{S}$. But $\#\mathcal{S}' = \#\mathcal{S}$ =10 so $\mathcal{S}' = \mathcal{S}$ as claimed. [In effect we have also shown that Π_5 satisfies the Pappus theorem: secants AB, CD, EF to an oval O meet at a point if and only if the corresponding three T points are collinear.]

6. To complete the determination of the structure of Π_5 we need only figure out which S points lie on which passant lines. We claim that an S point is on a passant line l if and only if the corresponding synthemes share exactly one pair. Indeed the three T points on l are the intersections of l with $3 \cdot 2 = 6$ secants; the three S points partition the remaining $15 - 2 = 9$ secants into 3 synthemes. Show that this can be done uniquely, with the synthemes claimed above. (Probably the simplest way is to actually draw the 9 secants.)

7. We conclude that Π_5, if it exists, is determined by a choice of total $\bar{\mathcal{S}}$ on a 6-element set O. Since the six totals on a 6-element set are equivalent, it follows that Π_5 is unique up to isomorphism if it exists — which it does because we already know a Π_5, namely the algebraic projective plane $\mathbf{P}^2(\mathbf{F}_5)$. Furthermore the number of automorphisms is the number 3100 of ovals times the number $6!/6 = 120$ of permutations of O preserving a total. Check that this is the same as the number of known $\mathrm{PGL}_3(\mathbf{F}_5)$ automorphisms of Π_5. Therefore $\mathrm{PGL}_3(\mathbf{F}_5)$ is the full automorphism group of Π_5. QED