Math 2^8: The Theory of Error-Correcting Codes

Positivity of N_{m+1} for extremal enumerators

Given power series $F = F_0^t = 1 + O(q)$ and $\Delta = q + O(q^2)$ and an integer $a \in [0, t)$, for every positive integer m there is a unique power series $W_m = 1 + O(q^{m+1})$ of the form $W_m = F_0^a P(F, \Delta)$ where P_m is a homogeneous polynomial of degree m. We are interested in the coefficients $N_{m+1}, N_{m+2}, \ldots$ of W_m. We begin with the following more-or-less elementary formula for N_{m+1}:

Lemma 1. *N_{m+1} is $(tm+a)/(tm+t)$ times the q^{-1} coefficient of $F_0^{-a}F'/\Delta^{m+1} = aF_0'F_0^{t-1-a}/\Delta^{m+1}$, where $F' = dF/dq$.*

For example, for the extended binary Golay code $\mathcal{G}_{24}$ we have $t = 3$, $F_0 = 1 + 14q + q^2$, $a = 0$, and $m = 1$, and calculate that

$$\frac{F_0^{-a}F'(q)}{\Delta^{m+1}} = \frac{(42+6q)F_0^2}{\Delta^2} = 42q^{-2} + 1518q^{-1} + 19452 + 117828q + \cdots$$

and indeed $\frac{3}{6}1518 = 759$.

Proof of Lemma 1: Let $z = \Delta/F = q + O(q^2)$. Then $W_m/(F_0^a F^m) = P(1, z)$, so if we let

$$\frac{1}{F_0^a F^m} = \sum_{k=0}^{\infty} b_k z^k$$

be the formal expansion of $1/(F_0^a F^m)$ in powers of z then $P(1, z) = \sum_{k=0}^{m} b_k z^k$. Therefore

$$\frac{1 - W_m}{F_0^a F^m} = \sum_{k=m+1}^{\infty} b_k z^k,$$

and in particular N_{m+1} is just $-b_{m+1}$. We isolate this coefficient using *invariance of the residue* under locally invertible change of coordinate; that's a result that's usually introduced in complex analysis but it can be proved also by formal power-series manipulation.[1] We find that $-b_{m+1}$ is

$$-\operatorname{Res}\left(\frac{1}{F_0^a F^m}\,\frac{dz}{z^{m+2}}\right) = -\operatorname{Res}\left(\frac{(\Delta/F)^{-m}}{F_0^a F^m}\,\frac{d(\Delta/F)}{(\Delta/F)^2}\right) = +\operatorname{Res}\frac{\Delta\,dF - F\,d\Delta}{F_0^a \Delta^{m+2}}.$$

But the $F\,d\Delta$ and $\Delta\,dF$ parts of this formula are related, because $F_0^a = F^{a/t}$ and $d(F^{1-(a/t)}\Delta^{-(m+1)})$ is an exact differential, and thus has residue zero:

$$\operatorname{Res}\frac{(1-\frac{a}{t})\Delta\,dF - (m+1)F\,d\Delta}{F_0^a \Delta^{m+2}} = 0.$$

Eliminating the $F\,d\Delta$ term and writing $dF = F'(q)\,dq$, we obtain our Lemma. $\diamondsuit$

Now in each of the cases where we want to find N_{m+1}, the power series F_0 is an enumerator and thus has nonnegative coefficients, while $1/\Delta$ has an expansion $q^{-1} + \sum_{k=0}^{\infty} c_k q^k$ in which we prove that every coefficient c_k is positive by writing $1/\Delta$ as a product of geometric series. Since $a \le t - 1$, it follows that $F_0'F_0^{t-1-a}/\Delta^{m+1}$ has positive q^k coefficient for all $k \ge -(m+1)$; in particular the q^{-1} coefficient is positive, so $N_{m+1} > 0$.

[1] Over a field K of characteristic zero we have an exact sequence

$$0 \to K \to K((q)) \to \Omega^1 K((q)) \to K \to 0,$$

where the map $K((q)) \to \Omega^1 K((q))$ is the differential and the map $\Omega^1 K((q)) \to K$ is the residue. A change of variable $z = z_1 q + O(q^2)$ with $z_1 \in K^*$ preserves everything except possibly the residue map, but then that map must be preserved up to some nonzero scalar, and since dz/z has the same residue as dq/q that scalar must be 1.