Math 250a: Higher Algebra
Problem Set #9 (6 December 2004):
Quaternion algebras

A bit more about Baer multiplication:

1. Let A be an abelian group and G any group acting on A. For any extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ consistent with this action, let E^o be the extension $1 \to A \xrightarrow{-\iota} E \xrightarrow{\pi} G \to 1$ with the opposite embedding of A in E. [Why do E, E^o have the same G-action on A?] Prove that E^o is the inverse of E in two ways: by identifying $(E, E^o)/Q$ with the semidirect product $A \rtimes G$, and by showing that E, E^o correspond to inverse elements of $H^2(G, A)$.

Note that the formula for E^o is what one might expect from the special case $(G, A) = (\mathrm{Gal}(L/k), L^*)$ and our results about the opposite of a central simple algebra.

In the next two problems, we describe generalized quaternion algebras over an arbitrary field k not of characteristic 2.

2. i) For any (commutative) field k, define a map $x \mapsto \bar{x}$ on $M_2(k)$ by $\bar{x} = \mathrm{Tr}(x) \cdot \mathbf{1} - x$. Here $\mathrm{Tr}(x)$ is the trace of x as a 2×2 matrix, and $\mathbf{1}$ is the 2×2 identity matrix, which is the unit element of $M_2(k)$. Prove that this map is an anti-involution, i.e., that it satisfies the identities $\bar{\bar{x}} = x$ and $\overline{xy} = \bar{y}\bar{x}$. [This can be done either by explicit computation or via a relation between $\bar{x}$ and the transpose of x.]
   ii) Now suppose that A/k is any central simple algebra with $\dim_k A = 4$. Define a map $x \mapsto \bar{x}$ on A by $\bar{x} = \mathrm{Tr}(x) \cdot 1 - x$, where $\mathrm{Tr}(x)$ is the reduced trace of x and 1 is the unit element of A. Prove that this map is an anti-involution.

Let A_0 be the kernel of Tr; it is a k-vector subspace of A of dimension $4-1 = 3$. Let $N : A \to k$ be the reduced norm, so $N(x) = x\bar{x}$. This is a quadratic form on A, and the associated bilinear form is

$$(x, y) = N(x + y) - N(x) - N(y) = x\bar{y} + y\bar{x} = \mathrm{Tr}(x\bar{y}).$$

Note that if $x \in A_0$ then $N(x) = -x^2$.

3. i) Prove that if $x, y \in A_0$ with $N(x) = N(y) \neq 0$ then x, y are conjugate in A.
   ii) Prove that there exist $\mathbf{i} \in A_0$ with $N(\mathbf{i})$ nonzero. Fix one such $\mathbf{i}$, and let $c = N(\mathbf{i})$. Since also $N(-\mathbf{i}) = N(\mathbf{i})$, by part (i) there exist invertible $z \in A$ such that $\mathbf{i}z = -z\mathbf{i}$. Show that $\mathbf{i}\bar{z} = -\bar{z}\mathbf{i}$, and hence that $\mathbf{ij} = -\mathbf{ji}$ where $\mathbf{j} := z - \bar{z}$. Show that $\mathbf{j} \in A_0$ and $\mathbf{j} \neq 0$.
   iii) Now let $\mathbf{k} = \mathbf{ij} = -\mathbf{ji}$. Show that $\mathbf{k} \in A_0$ and $\mathbf{ki} = -\mathbf{ik} = c\mathbf{j}$. Let $d = N(\mathbf{j}) = -\mathbf{j}^2$, and determine $\mathbf{jk}, \mathbf{kj}, \mathbf{k}^2$ in terms of $c, d, \mathbf{i}, \mathbf{j}, \mathbf{k}$. In particular show that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are pairwise orthogonal for the bilinear form $(\cdot, \cdot)$.
   iv) If A is a division algebra, prove that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are linearly independent, and

thus that $A = k + k\mathbf{i} + k\mathbf{j} + k\mathbf{k}$. What happens if $A = M_2(k)$?

v) Since we know the multiplication table of $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, we have determined A. Show that for any nonzero c, d the algebra obtained in this way is a division ring if and only if there are no $(r, s, t) \in k^3$ such that $cr^2 + ds^2 + cdt^2 = 0$ other than $(r, s, t) = (0, 0, 0)$.

It can be shown that every nondegenerate quadratic form on k^3 is equivalent to a multiple of $cr^2 + ds^2 + cdt^2 = 0$ for some $c, d \in k^*$; these c, d are not uniquely determined by the form, but the central simple algebras A associated to the quadratic form is uniquely determined by the equivalence class of the quadratic form up to scaling, and vice versa. Starting from part (i) we can also identify $A^*/\{\pm 1\}$ with the group of k-linear transformations of A_0 of determinant 1 that preserve the bilinear form $(\cdot, \cdot)$. This generalizes the identification of $\mathbf{H}^*/\{\pm 1\}$ with $\mathrm{SO}_3(\mathbf{R})$. If we regard $cr^2 + ds^2 + cdt^2 = 0$ as a conic in the projective plane over k, we get the simplest example of a "Brauer-Severi variety" associated to a central simple algebra.

If $k = \mathbf{R}$ and A is a division ring, then clearly $c, d > 0$; we may then scale $\mathbf{i}, \mathbf{j}$ by $c^{1/2}, d^{1/2}$ to identify A with $\mathbf{H}$. This completes the cohomology-free proof that $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$ are the only division algebras of finite dimension over $\mathbf{R}$. Likewise it can be shown that for each p there is a unique division algebra $\mathbf{H}_p$ with center $\mathbf{Q}_p$ and of dimension 4 over $\mathbf{Q}_p$. For odd p we constructed $\mathbf{H}_p$ in the seventh problem set. For our final problem, we treat the even case:

4. Find $c, d \in \mathbf{Q}_2^*$ that yield a division ring $\mathbf{H}_2$ with center $\mathbf{Q}_2$ and of dimension 4 over $\mathbf{Q}_2$.

You won't have to look very long for suitable c, d!

Problems 1–4 are due in class Monday, December the 13th.

5. Send me e-mail, or schedule a time to meet with me, to discuss your final paper topic.