Math 250a: Higher Algebra
Problem Set #8 (29 November 2004):
Tensor products etc.; Chevalley's Theorem

Some overdue basics on tensor products of algebras:

1. i) Let A be any algebra over a field k. Prove that $A \otimes_k M_n(k)$ is isomorphic as a k-algebra with $M_n(A)$, for each positive integer n.
   ii) Prove that the tensor product $M_m(k) \otimes_k M_n(k)$ is isomorphic as a k-algebra with $M_{mn}(k)$ for all positive integers m, n (and an arbitrary commutative field k).

An instructive example of a division algebra in positive characteristic:

Let k be a finite field, and A the noncommuting k-algebra generated by two indeterminates x, y satisfying $xy - yx = 1$.

2. i) Prove that A has no zero divisors, and that its center consists of the polynomials in $X := x^p$ and $Y := y^p$. (Hint for the second part: compare xy^a with $y^a x$, and likewise $x^b y$ with yx^b.)
   ii) Let $R = k[X, Y]$ be that center, and K its fraction field (the field of rational functions in two variables with coefficients in k). Let $D = A \otimes_R K$. Show that D is a skew field with center K.
   iii) We know that D must have a separable decomposition field. Find an *inseparable* field extension K'/K of degree p such that $D \otimes_K K' \cong M_p(K')$.

Can you find, at least for $p = 2$ or $p = 3$, an explicit isomorphism by finding endomorphisms x, y of K'^p satisfying $x^p = X$, $y^p = Y$, and $xy - yx = 1$?

Finally, we obtain Chevalley's theorem on solutions of polynomial equations in many variables over a finite field, and deduce the triviality of $\mathrm{Br}(k)$.

Fix a finite field $k = \mathbf{F}_q$, and let p be its characteristic, so q is some power of p.

3. i) Prove that $\sum_{x \in k} x^m = 0$ for all nonnegative integers $m < q - 1$. What is $\sum_{x \in k} x^m$ for an arbitrary nonnegative $m \in \mathbf{Z}$?

It follows that $\sum_{x \in k} P(x) = 0$ for any polynomial $P \in k[X]$ of degree less than $q - 1$. We next extend this to polynomials in several variables $X_1, \ldots, X_n$. The "degree" of a nonzero monomial $c \prod_{i=1}^n X_i^{m_i}$ is $\sum_i m_i$; the degree of a sum of distinct monomials is the largest of those monomials' degrees. This defines the degree on $k[X_1, \ldots, X_n]$. (Note that this degree is invariant under an invertible linear change of variables; thus we may speak of a polynomial of degree m on an n-dimensional vector space over k without specifying which coordinates on that space we use.)

ii) Prove that
$$\sum_{(x_1, \ldots, x_n) \in k^n} P(x_1, \ldots, x_n) = 0$$
for every polynomial $P \in k[X_1, \ldots, X_n]$ of degree less than $n(q - 1)$.

iii) Now let $f \in k[X_1, \ldots, X_n]$ be a polynomial of degree less than n. Take $P = f^{q-1}$ in (ii) to prove that the number of solutions in k^n of the equation $f(x_1, \ldots, x_n) = 0$ is a multiple of p.

As noted class the degree bound in (iii) is sharp. As also noted, it follows from (iii) that every finite skew field is commutative. Indeed, let K be such a field, and k its center. Then K is a vector space over k, of dimension n^2 for some positive integer n. The reduced norm is a polynomial of degree n on that vector space that vanishes only at the origin. Hence $n \geq n^2$. Therefore $n = 1$, and $K = k$ as claimed.

Can you generalize Chevalley's theorem to simultaneous solutions of several polynomials of low degree? Can you get a formula for the enumeration mod p of solutions of $x^3 + y^3 + z^3 = 0$ and $x^4 + y^4 = z^2$ in k^3, or $x^3 - x = y^2$ in k^2?

Problems 1–3 are due in class Monday, December the 6th.

4. Send me e-mail, or schedule a time to meet with me, to discuss your final paper topic.