Math 250: Higher Algebra
Problem Set #7 (12 November 2004):
Simple algebras, etc.

1. i) Fix a (commutative) field k. Is the (commutative, infinite-dimensional) algebra $k[X]$, considered as a module over itself, semisimple?
   ii) Now assume that k is not of characteristic 2. For each $c \in k$, let A_c be the (two-dimensional, commutative) k-algebra $k[X]/(X^2 - c)$. Prove that A_c, considered as a module over itself, is simple unless c is a square in k, and semisimple unless $c = 0$.
   iii) More generally, for any nonconstant polynomial $f \in k[X]$, when is the commutative k-algebra $k[X]/(f)$ simple or semisimple as a module over itself?

2. Prove that if M is a semisimple R-module then so is any submodule and any quotient module of M. Give an example of a ring R with a module M and a submodule M_0 such that M_0 and M/M_0 are both semisimple but M is not. [Hint: we've seen only a few examples of modules that are not semisimple. . .]

Linear algebra over skew fields. In our noncommutative setting, all modules, ideals, etc. will be left modules/ideals/. . . unless otherwise specified. A module over a skew field K is also called a "vector space" over K. This generalizes the notion of a vector space over a commutative field. We can use our generalities about semisimple modules to extend the basic notions of linear algebra to this setting.

3. i) Show that any skew field K, considered as a left K-module, is simple, and that every simple K-module is isomorphic to K.

Let V be a finitely generated[1] K-module, with generators $v_1, \ldots, v_n$. We may assume that each $v_i \neq 0$. Then V is the sum of the simple modules Kv_i, and is thus semisimple. Therefore it is the direct sum of Kv_j with j running over some subset $J \subseteq \{1, 2, \ldots, n\}$. If $V = \oplus_{j \in J} K v_j$, we say that $\{v_j : j \in J\}$ is a <u>basis</u> for V. Note that V is isomorphic as a K-module to K^m, where $m = \#(J) \leq n$. If $W \subseteq V$ is any K-submodule (also called "[K-vector] subspace"), then $V = W \oplus W'$ for some subspace $W' \subseteq V$, and W' can be chosen to be the direct sum of Kv_j with j running over some subset of J. In particular, if $\{v_j\}$ is a basis of V, and $\{u_i\}$ is any independent subset (that is, a subset such that the sum of Ku_i in V is direct), then $\{u_i\}$ can be completed to a basis.

ii) Prove that every linearly independent subset of K^n has cardinality at most n, and is a basis if and only if its cardinality is exactly n.

Thus the basis cardinality is an invariant of finitely generated K-vector spaces. Naturally we call this invariant the <u>dimension</u> of the space, and denote it by $\dim_K$.

iii) Let V be a K-vector space of finite dimension n, and T a K-linear map (that is, a K-module homomorphism) from V to some other K-vector space W. Prove that the dimensions of the image and kernel of T sum to n.

[1] We can extend the results of Problem 3 to arbitrary K-modules if we assume AC/Zorn.

We can also generalize duality of vector spaces to this non-commutative setting, provided we keep our directions straight:

4. Let K be a skew field, and K^o its opposite. For any K-vector space V, let V^* be its dual space, consisting of K-linear maps to K. Show that V^* has the structure of a K^o-vector space, and that if V has dimension $n < \infty$ then V^* is also n-dimensional as a K^o-vector space. Show further that, for any subspace $W \subseteq V$, its annihilator $\{v^* \in V^* : \forall w \in W, v^*(w) = 0\}$ is a K^o-subspace of V^* of dimension $\dim_K V - \dim_K W$.

5. Let K be a skew field, V a finite-dimensional vector space over K, and A its algebra of K-endomorphisms, isomorphic with the algebra of $n \times n$ matrices with entries in K. Show that for any $W \subseteq V$ the annihilator of W in A is a left ideal of A, and that two such ideals are isomorphic as representations of A if and only if they come from W's of the same dimension. (In particular, this proves and extends our claims about minimal left ideals, which come from W's of dimension $(\dim_K V) - 1$.) Show that every ideal of A is the annihilator of some W.

[You might first review how to do these problems in the familiar setting of vector spaces of commutative fields, and then extend to the noncommutative case. For the last part of Problem 5, you might begin from a recipe for recovering W from its annihilator, and then apply the recipe to an arbitrary left ideal. What are the right ideals of A?]

A p-adic analogue of the Hamilton quaternions. For each odd prime p we shall construct a 4-dimensional algebra $\mathbf{H}_p$ over the field $\mathbf{Q}_p$ of p-adic numbers, and prove properties analogous to those of the Hamilton quaternions over $\mathbf{R}$.

6. Fix an odd prime p, and a "quadratic nonresidue" s; that is, s is an integer not congruent mod p to m^2 for any $m \in \mathbf{Z}$. Let $\mathbf{H}_p$ be the $\mathbf{Q}_p$ algebra with generators $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ with the following multiplication table:

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	s	$\mathbf{k}$	$s\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	p	$-p\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-s\mathbf{j}$	$p\mathbf{i}$	$-sp$

For instance, $\mathbf{ij} = \mathbf{k}$. Define an involution $x \leftrightarrow \bar{x}$ on $\mathbf{H}_p$ by $\bar{x} = 2a - x$ where $x = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. The (reduced) norm $N : \mathbf{H}_p \to \mathbf{Q}_p$ is defined by

$$N(x) = x\bar{x} = \bar{x}x = a^2 - sb^2 - pc^2 + spd^2.$$

i) Verify that $\mathbf{H}_p$ is associative.
ii) Verify that $x \leftrightarrow \bar{x}$ is an isomorphism from $\mathbf{H}_p$ to its opposite algebra $\mathbf{H}_p^o$. (Such a map is called an "anti-isomorphism" of $\mathbf{H}_p$, and an "anti-involution" if, as here, it is its own inverse.) Use this to show that the norm map is multiplicative: $N(xy) = N(x)N(y)$ for all $x, y \in \mathbf{H}_p$.
iii) Show that $N(x) = 0$ if and only if $x = 0$, and conclude that $\mathbf{H}_p$ is a skew field.

[If you're not used to working with p-adic numbers, you can interpret the first part of (iii) as "show that $a^2 - sb^2 - pc^2 + spd^2 = 0$ has no nonzero rational solutions $(a, b, c, d) \in \mathbf{Q}^4$ using only congruences modulo powers of p".]

Problem set is due in class Monday, November the 22nd.