Math 250: Higher Algebra

Problem Set #5 (29 October 2004): Representations of finite groups

First, two exercises from Fulton-Harris on induced representations. Let G be a finite group, and H any subgroup.

1. [Fulton-Harris, Exercise 3.16 on page 34]
   i) If U is a representation of G and W is a representation of H, show that
   $$U \otimes \mathrm{Ind} W = \mathrm{Ind}(\mathrm{Res}(U) \otimes W).$$
   In particular, $\mathrm{Ind}(\mathrm{Res}(U)) = U \otimes P$, where P is the permutation representation of G on G/H.
   ii) If $H \subset K \subset G$ with K also a subgroup, show that
   $$\mathrm{Ind}_H^G W = \mathrm{Ind}_K^G(\mathrm{Ind}_H^K W)$$
   (so induction, like restriction, is transitive).

2. [Fulton-Harris, Exercise 3.19 on page 34] Let W be a finite-dimensional representation of H, and $V = \mathrm{Ind}_H^G W$. Quoting Fulton-Harris (p.34):

   > To compute [its] character, note that $g \in G$ maps σW to $g\sigma W$, so the trace is calculated from those cosets σ with $g\sigma = \sigma$, i.e., $s^{-1}gs \in H$ for [any, equivalently all] $s \in \sigma$. Therefore,
   > $$\chi_{\mathrm{Ind}\, W}(g) = \sum_{g\sigma=\sigma} \chi_W(s^{-1}gs) \qquad (s \in \sigma \text{ arbitrary}). \tag{3.18}$$

   i) If C is the conjugacy class of g in G, and $C \cap H$ decomposes into conjugacy classes $D_1, \ldots, D_r$ of H, then (3.18) can be rewritten as:
   $$\chi_{\mathrm{Ind}\, W}(g) = [G : H] \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi_W(h_i)$$
   for $h_i \in D_i$.
   ii) If W is the trivial representation of H, then
   $$\chi_{\mathrm{Ind}\, W}(g) = \frac{[G : H]}{|C|} |C \cap H|.$$
   iii) Deduce the Frobenius reciprocity formula (Cor. 3.20 on p.35): for any finite-dimensional representations U, W of G, H respectively,
   $$(\chi_{\mathrm{Ind}\, W}, \chi_U)_G = (\chi_W, \chi_{\mathrm{Res}\, U})_H.$$

We next describe "extraspecial 2-groups" and their representations. The 8-element dihedral and quaternion groups are the familiar first examples of extraspecial 2-groups.

3. Let G be a finite group whose center contains a 2-element group $\{\pm 1\}$, and let A be the quotient group $G/\{\pm 1\}$. Let V be any irreducible representation of G, of dimension d. Show that the restriction of V to $\{\pm 1\}$ consists of either d copies of the trivial representation — in which case it comes from a representation of A — or d copies of the nontrivial 1-dimensional representation of $\{\pm 1\}$.

   Equivalently, $-1 \in G$ acts on V by multiplication by a scalar, which is necessarily either $+1$ or -1. We'll call these two kinds of representations of G "even" and "odd" respectively.

4. Now suppose A is an "elementary abelian 2-group", that is, a group isomorphic with $(\mathbf{Z}/2\mathbf{Z})^m$ for some m. Define a map $(\cdot,\cdot) : A \times A \to \{\pm 1\}$ as follows: for any $a, b \in A$, let $g \in G$ be either of the preimages of a, and let $h \in G$ be either of the preimages of b; then (a, b) is the commutator $ghg^{-1}h^{-1}$. Explain why (a, b) is in fact in $\{\pm 1\}$ and is well-defined (independent of the choice of g, h). Then show that $(\cdot,\cdot)$ is bilinear and alternating, i.e., that it satisfies the identities

$$(aa', b) = (a, b)(a', b), \quad (a, bb') = (a, b)(a, b'), \quad (a, a) = 1, \quad (a, b) = (b, a).$$

   Prove that this pairing is nondegenerate if and only if $\{\pm 1\}$ is the center of G, and if and only if each $g \in G - \{\pm 1\}$ is conjugate to $-g = (-1)g$. In this case, G is said to be an "extraspecial 2-group". Note that the 8-element dihedral and quaternion groups are indeed extraspecial 2-groups with $m = 2$.

5. Now let G an extraspecial 2-group of order 2^{m+1}.
   i) Show that G has 2^m even representations, each of dimension 1. Deduce that G has a unique odd representation V and that V has dimension $2^{m/2}$. [In particular it follows that m must be even — which we could also obtain from the nondegeneracy of the alternating form $(\cdot,\cdot)$.] Determine the character χ_V of this representation, and note that $\chi_V(g) \in \mathbf{R}$ for all $g \in G$, whence V is necessarily either real or quaternionic.
   ii) For $a \in A$ define $Q(a) = g^2$ where g is either of the preimages of a in G. Explain why this gives a well-defined map from A to $\{\pm 1\}$, and show that it is a quadratic form whose associated bilinear form is $(\cdot,\cdot)$ (which, in our multiplicative notation for the G and its center, means that $(a, b) = Q(ab)/(Q(a)Q(b))$ for all $a, b \in A$). Use the formula of Exercise 3.38* (p.41) to conclude that V is real or quaternionic according as $\sum_{a \in A} Q(a)$ equals $2^{m/2}$ or $-2^{m/2}$.

Note that this confirms the known behavior of the 8-element extraspecial 2-groups. It turns out that any quadratic form on $(\mathbf{Z}/2\mathbf{Z})^m$ comes from some extraspecial 2-group, and thus in particular satisfies $\sum_{a \in A} Q(a) = \pm 2^{m/2}$; and that two quadratic forms on $(\mathbf{Z}/2\mathbf{Z})^m$ are equivalent under $\mathrm{Aut}((\mathbf{Z}/2\mathbf{Z})^m) = \mathrm{GL}_{2m}(\mathbf{Z}/2\mathbf{Z})$ if and only if their invariants $2^{-m/2} \sum_{a \in A} Q(a)$ are equal. It follows that for each positive integer m there are two extraspecial 2-groups of order 2^{m+1}, usually denoted 2_+^{1+m} and 2_-^{1+m}. For instance, the 8-element dihedral and quaternion groups are 2_+^{1+2} and 2_-^{1+2}.

Problem set is due in class Friday the 5th of November.