Math 155: Designs and Groups

Homework Assignment #8 (12 April 2010):
$A_7 \hookrightarrow A_8 \cong \mathrm{GL}_4(\mathbf{F}_2)$ cont'd; finite subgroups of $\mathrm{PSL}_2(F)$ and related matters

This problem set is due Monday, April 19 in class.

1. Let Σ be an unstructured 7-element set. We showed that the action of A_7 on the 30 Π_2-structures on Σ has two orbits of size 15, each of which may be identified with the set nonzero vectors in a 4-dimensional vector space over $\mathbf{F}_2$. Let V, W be these two spaces. So far we concerned ourselves with one of these; we now explore their interrelationship.

   If P, Q are Π_2 structures in different A_7 orbits, show that they have either no lines in common or three lines through a single point. Moreover, given P and a point $p \in \Sigma$ there is a unique Q sharing with P all three lines through p.

   For $P \in V$, $Q \in W$ we now define $\langle P, Q \rangle \in \mathbf{F}_2$ as follows: If either P or Q is zero, $\langle P, Q \rangle = 0$; else $\langle P, Q \rangle$ is 0 if the Π_2-structures P, Q share three lines, and 1 if they have no lines in common. Show that for each nonzero $Q \in W$ the set $\{P \in V : \langle P, Q \rangle = 0\}$ is a subspace of V of codimension 1. Conclude that $\langle \cdot, \cdot \rangle$ is a nondegenerate pairing, so naturally identifies each of V, W with the other's dual.

2. Let G be the group with the following presentation by generators and relations:

$$G = \langle b, c, d \mid b^2 = c^4 = d^4 = bcd = 1 \rangle.$$

   Construct the Cayley graph of G with respect to $\{c, d\}$, and use this to identify G with a subgroup of $\mathrm{AGL}_2(\mathbf{R})$.

   [There is a similar description for the groups constructed in the same way with the exponents $(2, 4, 4)$ replaced by the other solutions $(3, 3, 3)$ and $(2, 3, 6)$ of $1/e_1 + 1/e_2 + 1/e_3 = 1$, but it's a bit harder to describe those because we have more experience with square grids than we do with triangular or hexagonal ones.]

3. i) Show that the image in $\mathrm{PGL}_2(F)$ of a 2×2 matrix $A \in \mathrm{GL}_2(F)$ has order 4 if and only if $\mathrm{tr}(A)^2 = 2\det(A)$ but $\mathrm{tr}(A) \neq 0$. Use this to prove that if F is finite then $\mathrm{PGL}_2(F)$ contains S_4 if and only if F is not of characteristic 2.

   ii) Prove on the other hand that if $\mathrm{PSL}_2(F)$ does not contain A_5 for some field F then $\mathrm{PGL}_2(F)$ does not contain A_5 either. (Hint: you don't need to figure out the analogue of part (i) for elements of order 5.)

4. Prove that $\mathrm{PGL}_2(\mathbf{R})$ does not contain A_4 (and thus does not contain S_4 or A_5 either, even though it certainly has elements of order n for any n).

5. Check directly that the centralizer of every non-identity element of A_4 or A_5 is abelian. Show however that this does not hold for S_4. Why does this not indicate a flaw in the proof of the classification of finite subgroups of $\mathrm{PSL}_2(F)$?