Math 155: Designs and Groups

Homework Assignment #7 (31 March 2010):
The Tutte 8-cage; transvections; about the representation $A_7 \hookrightarrow A_8 \cong \mathrm{GL}_4(\mathbf{F}_2)$

This problem set is due Friday, Apr.9 in class.

1. The *Tutte 8-cage* is the bipartite graph T whose vertices are labeled by the pairs and synthemes of a 6-element set, with each syntheme adjacent to its component pairs. Show that T has girth 8 and that no cubic graph of girth 8 can have less than T's 30 points. (It is known that T is the unique cubic graph of girth 8 on 30 vertices but you need not prove this.)
Show that $\mathrm{Aut}(T)$ is the 1440-element group $\mathrm{Aut}(S_6)$. [This is easier than it may seem once you note that $\mathrm{Aut}(T)$ contains that group. Every automorphism either preserves the two parts of T or switches them; it is enough to prove that every automorphism preserving the parts is in S_6. Do this by relating each part with the triangle graph $T_2(6)$. Cf. problem 1a of the fifth homework set.]

2. Show that T can also be described thus: the vertices are the blocks of the 3-(10,4,1) design, each of which is adjacent to the three blocks disjoint from it.

3. i) Let k be a field of characteristic 2. Show that the only involutions in $\mathrm{PGL}_3(k)$ are the transvections. For $n > 3$ find an involution in $\mathrm{PGL}_n(k)$ that is not a transvection.
ii) Prove that the finite simple group $\mathrm{PSL}_3(\mathbf{F}_4)$ contains no element of order 6, and thus is not isomorphic with the group $\mathrm{GL}_4(\mathbf{F}_2) \cong A_8$ of the same size.

For the last two problems: since $\mathrm{GL}_4(\mathbf{F}_2) \cong \mathrm{A}_8$ there is an index-8 subgroup of $\mathrm{GL}_4(\mathbf{F}_2)$ isomorphic with A_7. In other words, A_7 acts on the 15 nonzero vectors in $\mathbf{F}_2^4$. It turns out that the action is doubly transitive. In particular the point stabilizer has size $\#(A_7)/15 = 168$, which is suggestive...

4. Let Σ be an unstructured 7-element set. A "Π_2-structure" is a choice of 2-(7,3,1) design of subsets of Σ (i.e. an identification of Σ with the points of Π_2). Show that there are 30 such structures, and that the action of A_7 on them has two orbits of size 15. Give a combinatorial definition, similar to what we did for hyperovals and subplanes of Π_4, of an equivalence relation on the Π_2-structures for which the equivalence classes are the two A_7 orbits; and prove directly that it is an equivalence relation.

5. Let P be one of the orbits. Since we expect to identify P with the nonzero vectors of $\mathbf{F}_2^4$, there should be a distinguished set of three-element subsets of P, namely the sets of nonzero vectors in 2-dimensional subsets of $\mathbf{F}_2^4$, which are the blocks of a $(2, 3, 15)$ Steiner system. Give a combinatorial construction of such a Steiner system of subsets of P. Use this to provisionally define the structure of an $\mathbf{F}_2$ vector space on $P \cup \{0\}$. What must you check to verify that it works?

This can be used to give an alternative approach to the isomorphism $\mathrm{GL}_4(\mathbf{F}_2) \cong A_8$.