Math 155: Designs and Groups

Homework Assignment #6 (22 March 2010):
Some finite group theory

This short problem set is due Monday, March 29 in class.

1. Let G be a finite group and H a (necessarily) normal subgroup of index 2 in G. Prove that, of the G-conjugacy classes, each class disjoint from H is permuted transitively under conjugation by elements of H, while each class contained in H is either permuted transitively (and thus constitutes a conjugacy class of H) or splits into two H-conjugacy classes of equal size. For $(G, H) = (S_n, A_n)$, which cycle structures (i.e. which $(a_1, a_2, \ldots, a_n)$ with $\sum_i ia_i = n$) correspond to split conjugacy classes?
   [Hint: 3.22 on the Rotman handout may help.]

2. Verify assertions (1), (2), (3) on p.22 of the Cameron-van Lint textbook. That is, show that the group-theoretical partition of the 168 (hyper)ovals in Π_4 into three equal classes of 56 has the following combinatorial description: two ovals O, O' are in the same class iff $\#(O \cap O')$ is even. Do the same for the $3 \cdot 120 = 360$ Baer subplanes S (this time $S \sim S'$ iff $\#(S \cap S')$ is odd), and show that the parity of $\#(O \cap S)$ depends only on the classes of O, S, and given O or S is even for exactly one class of S's or O's. [There are various symmetries of Π_4, ovals, and subplanes that you can exploit to greatly reduce the number and complexity of cases you'll need to check.]

3. i) Find a 24-element subgroup G of $\mathrm{PGL}_2(\mathbf{F}_5)$ and use its existence to obtain a different proof of the isomorphism $\mathrm{PGL}_2(\mathbf{F}_5) \cong S_5$.
   ii) Find a 24-element subgroup G of $\mathrm{PSL}_2(\mathbf{F}_7)$ to obtain an embedding of $\mathrm{PSL}_2(\mathbf{F}_7)$ into A_7.
   [Hint: S_4 has presentation $\langle \sigma, \tau | \sigma^2 = \tau^3 = (\sigma\tau)^4 = 1 \rangle$.]