

## Math 155: Designs and Groups

Homework Assignment #5 (26 March 2010):  
Synthemes, Totals, and  $\Pi_5$

This problem set is due Friday, March 12 in class.

- Fix a 6-element set  $\mathbf{6}$ . Let  $G$  be the graph whose vertices are the 15 synthemes on  $\mathbf{6}$  with two synthemes adjacent if and only if they have a pair in common. Show (without invoking  $\text{Aut}(S_6)$ ) that  $G$  is strongly regular with the same parameters as the complement of the triangle graph  $T_2(\mathbf{6})$  whose vertices are the 2-element subsets of  $\mathbf{6}$ . Then use an outer automorphism of  $S_6$  to show that indeed the two graphs are isomorphic.
  - The maximal co-cliques of  $G$  are the totals (check that again we have equality in the  $-sn/(k-s)$  bound). What is the graph obtained from  $G$  by deleting such a co-clique and all its incident edges?
- The remaining exercises outline a proof of the uniqueness of the projective plane of order 5 and the identification of its automorphism group with  $\text{PGL}_3(\mathbf{F}_5)$ ; fill in the details. Recall first that an *oval*  $O$  in such a plane  $\Pi_5$  (necessarily of Type I since 5 is odd) is a set of 6 points with no three collinear. Begin by showing that the number of ovals in  $\Pi_5$  is  $31 \cdot 30 \cdot 25 \cdot 16 \cdot 6 \cdot 1/6! = 3100$ . Fix one oval  $O$ . Of the  $5^2 + 5 + 1 = 31$  lines of  $\Pi_5$ , six are tangent,  $\binom{6}{2} = 15$  are secant, and the remaining 10 passant to  $O$ .
- Of the  $31 - 6 = 25$  points of  $\Pi_5 - O$  let  $n_i$  be the number lying on  $i$  tangents (necessarily  $2|i$ ). As in Prop. 1.48 compute that  $\sum_{i>0} i n_i = 6 \cdot 5 = 30$  and  $\sum_{i>0} \binom{i}{2} n_i = \binom{6}{2} = 15$  to conclude that  $n_2 = 15$  and  $n_i = 0$  for all  $i > 2$ . Thus each pair of tangents meets in a different point; we call such a point a “tangential point” or T point for short. Each of the remaining  $n_0 = 25 - 15 = 10$  points of  $\Pi_5$  lies on no tangent, so is the intersection of three secants to  $O$ , determining a syntheme. We call such a point a “synthematic point” or S point, and let  $\mathcal{S}$  be the set of ten synthemes determined by the S points.
- Any secant meets the tangents to its endpoints on  $O$ , and the remaining  $6 - 2 = 4$  tangents off  $O$ . Thus it meets them in pairs at  $4/2$  or two T points. Thus the remaining  $6 - 2 - 2 = 2$  points on the secant are S points. It follows that each pair occurs twice in the  $\mathcal{S}$  synthemes, whence  $\mathcal{S}$  is the complement of a total.

5. Of the 6 lines through each T point, 2 are tangent and 2 secant to  $O$ , so the remaining 2 are passant. Thus the number of pairs  $P, l$  with  $P$  a T point and  $l$  a passant line through  $P$  is  $15 \cdot 2 = 30$ . But no passant line may go through more than three T points ( $3 = 6/2$ ): any more than that include two that lie on a common tangent. Thus each of the 10 passant lines contains exactly three T points (and thus three S points since  $6 - 3 = 3$ ).
6. Since T points correspond to pairs in  $O$ , each passant line determines via its T points three disjoint pairs, i.e. a syntheme. Let  $\mathcal{S}'$  be the set of ten synthemes thus arising. We'll show  $\mathcal{S}' = \mathcal{S}$ . Indeed if  $(AB)(CD)(EF)$  is a syntheme *not* in  $\mathcal{S}$  than the T point at the intersection of the  $A, B$  tangents lies on the  $CD, EF$  secants; likewise for  $C, D$ . Thus the line joining these two T points is the  $EF$  secant, which does not contain the T point  $E, F$ , whence  $(AB)(CD)(EF) \notin \mathcal{S}'$ . Thus  $\mathcal{S}' \subseteq \mathcal{S}$ . But  $\#\mathcal{S}' = \#\mathcal{S} = 10$  so  $\mathcal{S}' = \mathcal{S}$  as claimed. [In effect we have also shown that  $\Pi_5$  satisfies the Pappus theorem: secants  $AB, CD, EF$  to an oval  $O$  meet at a point if and only if the corresponding three T points are collinear.]
7. To complete the determination of the structure of  $\Pi_5$  we need only figure out which S points lie on which passant lines. We claim that an S point is on a passant line  $l$  if and only if the corresponding synthemes share exactly one pair (i.e. are adjacent on the graph of Problem 1b). Indeed the three T points on  $l$  are the intersections of  $l$  with  $3 \cdot 2 = 6$  secants; the three S points partition the remaining  $15 - 2 = 9$  secants into 3 synthemes. Show that this can be done uniquely, with the synthemes claimed above. (Probably the simplest way is to actually draw the 9 secants.)
8. We conclude that  $\Pi_5$ , if it exists, is determined by a choice of total  $\bar{S}$  on a 6-element set  $O$ . Since the six totals on a 6-element set are equivalent, it follows that  $\Pi_5$  is unique up to isomorphism if it exists — which it does because we already know a  $\Pi_5$ , namely the algebraic plane  $\mathbf{P}^2(\mathbf{F}_5)$ . Furthermore the number of automorphisms is the number 3100 of ovals times the number  $6!/6 = 120$  of permutations of  $O$  preserving a total. Check that this is the same as the number of known  $\text{PGL}_3(\mathbf{F}_5)$  automorphisms of  $\Pi_5$ . Thus  $\text{PGL}_3(\mathbf{F}_5)$  is the full automorphism group of  $\Pi_5$ . QED