Math 155: Designs and Groups

Homework Assignment #4 (17 February 2010):
Spherical 3-designs; regular graphs, cont'd

This problem set is due Friday, Feb. 26 in class.

1. i) Let V be a vector space of dimension 2 over a field F. Suppose W_1, W_2, W_3 are pairwise distinct 1-dimensional subsets of V, as are X_1, X_2, X_3 (but with no condition that W_i be distinct from X_j). Prove that there exists a linear transformation $T : V \to V$ such that $T(W_i) = X_i$ for each of $i = 1, 2, 3$, and that any such T is invertible. Given one such T, prove that a linear transformation $T' : V \to V$ also satisfies $T(W_i) = X_i$ for each of $i = 1, 2, 3$ if and only if $T' = cT$ for some nonzero scalar $c \in F^*$.

   ii) Now suppose k is a subfield of a field K, and regard $\mathbf{P}^1(k)$ as the subset $\{(x_0 : x_1) \mid x_0, x_1 \in k\}$ of $\mathbf{P}^1(K)$. Prove that any three distinct points in $\mathbf{P}^1(K)$ are contained in a unique subset of the form $g(\mathbf{P}^1(k))$ with g in $\mathrm{PGL}_2(K)$. [Hints: if g works then so does gh for any $h \in \mathrm{PGL}_2(k)$; in (i) you proved in effect that $\mathrm{PGL}_2(F)$ acts simply 3-transitively on $\mathbf{P}^1(F)$.]

2. [Asserted at the bottom of page 34 of the textbook] Prove that a strongly regular graph with $\mu = 0$ must consist of r disjoint copies of the complete graph K_m for some $r, m > 1$.

3. If A is the incidence matrix of a graph G, what (in terms of A, I, J) is the incidence matrix of its complement $\overline{G}$? Use the incidence-matrix formulation of strong regularity to give an alternative proof of Prop. 2.7 (the strong regularity and parameters of the complement of a strongly regular graph).

4–7. Solve problems #3,4,6,5 on page 45 of the textbook. [For the first of these, see pages 41 and 42 for the relevant bounds; for the last, note that Ex. 2.21 is two paragraphs long. . .]

8. A *clique* in a graph G is a set of vertices of G any two of which are adjacent; a *co-clique* of G is a clique of the complementary graph $\overline{G}$, i.e. a set of vertices no two of which are adjacent in G. Prove that in a regular graph G of degree k on n vertices, any co-clique contains at most $-sn/(k-s)$ vertices, where s is the smallest eigenvalue of the adjacency matrix of G. [NB: G is <u>not</u> assumed *strongly* regular. Necessarily $s < 0$ (why?)]

9. Find a co-clique of size $-sn/(k-s)$ in each of the following graphs: the square lattice graph $L_2(m)$, the graph $r \cdot K_m$ of problem 1, and the Petersen graph. How many such maximal co-cliques are there in each case?