

Math 155: Designs and Groups

Homework Assignment #2 (1 February 2010):
More about designs

This problem set is due Wednesday, Feb.10 in class.

1. i) In Theorem 1.20 (page 6), suppose that \mathcal{D} is not just a 2-design but even a 3-design. Give a formula for

$$\sum_i i^3 n_i = \sum_{B' \neq B} \#(B \cap B')^3$$

analogous to what we found for $\sum_i i n_i$ and $\sum_i i^2 n_i$. (Note that in the $\sum_i^2 n_i$ formula λ must now be replaced by λ_2 .)

- ii) Suppose now that \mathcal{D} is a 3-(22, 6, 1) design. Show that, for any block B' other than B , the intersection $B \cap B'$ is either empty or contains exactly 2 points. How many B' are disjoint from B ? (You may be able to do this without part (i), using only the formulas from Theorem 1.20. We shall see that there exists a 3-(22, 6, 1) design, and it is unique up to isomorphism, with an interesting automorphism group that contains the sporadic Mathieu group M_{22} with index 2.)
2. Prove Proposition 1.39 (the complement of a t -design is a t -design) for $t = 2$ using the incidence matrix and verify in that case that $\bar{\lambda} = b - 2r + \lambda$ [formula (1.38)].
3. For which q, n does the square 2-design of points and hyperplanes in $\mathbf{F}_q \mathbf{P}^n$ have an even number of points? Verify that the criterion of Bruck-Ryser-Chowla (Thm. 1.21a) holds for these designs.
4. [CvL p.25 #2] The Kronecker product or tensor product of matrices $A = (a_{ij})$ and $B = (b_{kl})$ is the matrix $A \otimes B$ with $((i, k), (j, l))$ entry $a_{ij} b_{kl}$. Prove that the Kronecker product of Hadamard matrices of orders n_1 and n_2 is a Hadamard matrix of order $n_1 n_2$. Prove also that the Kronecker product of a number of copies of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is a Sylvester matrix.
5. [CvL p.25 #3, extended] Suppose that H is a Hadamard matrix of order $n > 1$ with constant row and column sums d . Prove that $n = d^2$ with d even. Prove also that $(H + J)/2$ [i.e. H with its -1 entries replaced by zeros] is the incidence matrix of a 2- $(4u^2, 2u^2 + u, u^2 + u)$ design where $d = 2u$ (NB: d may be positive or negative). Show that the Kronecker product of two such H 's is again of the same form, and use this to construct a square 2-(16,6,2) design.
6. [CvL p.26 #10] Use the method of "intersection triangles" to prove that if Y is a set of 5 of the points of a 3-(10,4,1) design then Y contains no block if and only if the complement of Y contains no block.

7. [CvL p.25 #5] Let x_1, x_2, \dots, x_{t+1} be points of a t -(v, k, λ) design $\mathcal{D} = (X, \mathcal{B})$, and let μ be the number of blocks containing those $t + 1$ points. Use the Principle of Inclusion and Exclusion to show that there exists a number F , depending only on the parameters t, v, k, λ of \mathcal{D} , such that the number of blocks containing none of x_1, x_2, \dots, x_{t+1} is $F + (-1)^{t+1}\mu$.

Deduce that if $v = 2k + 1$ and t is even then

$$\mathcal{E} := (X \cup \{\infty\}, \{B \cup \{\infty\}, X - B : B \in \mathcal{B}\})$$

is a $(t + 1)$ -design extending \mathcal{D} .

8. Prove that the square 2-design coming from a perfect difference set (see problem 2 of the previous homework set) is self-dual and in fact admits a polarity.
9. [CvL p.26 #8a, extended] Prove that the Paley square 2-(11,5,2) design is the unique design with these parameters up to isomorphism. How large is its automorphism group?
10. [CvL p.26 #8b; Assmus-Mezzaroba-Salwach 1977] Let X be the set of points *and* blocks of the 2-(11,5,2) design. Let \mathcal{B} be the family of subsets of X of the following types:
- a point and the five blocks containing it;
 - a block and the five points contained in it;
 - an oval and its tangents.

Prove that (X, \mathcal{B}) is a 3-(22,6,1) design (a.k.a. $S(3, 6, 22)$ Steiner system).