Math 155: Designs and groups
Handout #5:
The 3-(8,4,1) Steiner system and the isomorphism $\mathrm{PSL}_2(\mathbf{F}_7) \cong \mathrm{GL}_3(\mathbf{F}_2)$

Let $\mathcal{D}$ be the 3-(8,4,1) Steiner system. It exists because for instance the affine planes in $\mathbf{F}_2^3$ give a suitable system of 14 blocks; and it is unique because the derived design is the unique Π_2, and from the intersection triangle of $\mathcal{D}$ we know that $\mathcal{D}$ contains the complement of each of its blocks, so that accounts for the $7+7=14$ blocks of $\mathcal{D}$.

Let $G = \mathrm{Aut}(\mathcal{D})$. This contains the affine linear group $\{v \mapsto Av+b : A \in \mathrm{GL}_3(\mathbf{F}_2), b \in \mathbf{F}_2^3\}$. In particular G permutes $\mathbf{F}_2^3$ transitively so its order is $8 \cdot \#\mathrm{Aut}(\Pi_2) = 8 \cdot 168$. Since that is the number of affine linear transformations, G is identified with that affine linear group. We obtain a surjective homomorphism $G \to \mathrm{GL}_3(\mathbf{F}_2)$ by mapping $v \mapsto Av+b$ to A. The kernel of this homomorphism is the 8-element group of translations $x \mapsto x+b$.

But we can also obtain $\mathcal{D}$ as follows: the 8 points are $\mathbf{P}^1(\mathbf{F}_7)$, and the blocks are the images under $\mathrm{PSL}_2(\mathbf{F}_7)$ of $\{0,1,3,\infty\}$. Since the stabilizer of this in $\mathrm{PSL}_2(\mathbf{F}_7)$ is A_4, there are $\#\mathrm{PSL}_2(\mathbf{F}_7)/\#A_4 = 168/12 = 14$ blocks. Since $\mathrm{PSL}_2(\mathbf{F}_7)$ acts transitively on 3-element subsets of $\mathbf{P}^1(\mathbf{F}_7)$ [even though it does <u>not</u> act 3-transitively![1]], the blocks constitute a 3-design. So we get a 3-(8,4,1) design with automorphisms by $\mathrm{PSL}_2(\mathbf{F}_7)$. Since our design must be isomorphic with $\mathcal{D}$, this means $\mathrm{PSL}_2(\mathbf{F}_7)$ is contained in G.

Composing the inclusion $\mathrm{PSL}_2(\mathbf{F}_7) \hookrightarrow G$ with the homomorphism $G \to \mathrm{GL}_3(\mathbf{F}_2)$ we obtain a homomorphism $\mathrm{PSL}_2(\mathbf{F}_7) \to \mathrm{GL}_3(\mathbf{F}_2)$ whose kernel has order at most 8. But $\mathrm{PSL}_2(\mathbf{F}_7)$ is simple of order 168 so the kernel is trivial. Since $\mathrm{GL}_3(\mathbf{F}_2)$ also has order 168 our map is in fact an isomorphism.

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[1] Since $\mathrm{PSL}_2(\mathbf{F}_7)$ is of index 2 in $\mathrm{PGL}_2(\mathbf{F}_7)$, we can prove this transitivity by showing that the stabilizer in $\mathrm{PGL}_2(\mathbf{F}_7)$ of a three-point set is not contained in $\mathrm{PSL}_2(\mathbf{F}_7)$. Since all three-point sets are equivalent under $\mathrm{PGL}_2(\mathbf{F}_7)$ we choose $\{0,1,\infty\}$ and note that the involution $x \leftrightarrow 1-x$ permutes it but is not in $\mathrm{PSL}_2(\mathbf{F}_7)$ because -1 is not a square. This argument shows that for any field F the group $\mathrm{PSL}_2(F)$ acts transitively on three-point subsets of $\mathbf{P}^1(F)$ if and only if -1 is not a square in F. When F is finite this means $|F|$ is not 1 mod 4. So the first counterexample is $\mathbf{F}_5$, when $\mathrm{PSL}_2(F)$ has two orbits on the 3-point subsets. Here the complement of such a subset is of the same size; is it in the same orbit?