Riemann surfaces, dynamics and geometry

Harvard University — Spring 1998 — Math 275
C. McMullen

Selected Solutions

Show any infinite Kleinian group Γ contains an element of infinite order.

We begin with a lemma on quasi-geodesics.

Consider an infinite sequence $P = \{p_i \ : \ i \in \mathbb{Z}\}$, $p_i \in \mathbb{H}^n$. Think of the p_i as the vertices of an infinite polygonal path with segments $[p_i, p_{i+1}]$. Suppose we have constants (L, α) such that the length of $[p_i, p_{i+1}]$ is at least L, and the angle at p_i of the triangle $[p_{i-1}, p_i, p_{i+1}]$ is at least $\alpha > 0$.

Lemma 0.1 *For each α, if L is large enough ($L > L(\alpha)$), then there is a unique geodesic $\gamma \subset \mathbb{H}^n$ contained in a bounded neighborhood of P.*

In other words, the polygonal path P approximately follows γ.

Sketch of the proof: Consider the perpendicular bisector H_i of each segment $[p_i, p_{i+1}]$. That is, H_i is the locus of points equidistance between p_i and p_{i+1}. Then H_i is a hyperplane meeting S^{n-1}_∞ in an $(n-2)$-sphere S_i. If L is large enough, these spheres S_i are nested with diameters tending to zero. As $i \to \pm\infty$, the spheres converge to the two endpoints of γ. ■

We can use this lemma to recognize hyperbolic elements g in a Kleinian group Γ. Namely if the angle between the segments $[p, g(p)]$ and $[g(p), g^2(p)]$ is α and $d(p, g(p)) > L(\alpha)$ (the constant given by the lemma), then $P = \langle g^i(p)\rangle$ shadows a geodesic γ stabilized by g, and g is hyperbolic.

In terms of the quotient orbifold $M = \mathbb{H}^n/\Gamma$, the segment $[p, g(p)]$ projects to a geodesic loop representing g in $\pi_1(M, [p])$. The angle α is nothing more than the angle this loop makes at $[p]$ (the projection of p to M).

Solution to the problem. Let Γ be an infinite Kleinian group. Then the limit set Λ is nonempty, and the case where $|\Lambda| = 1$ is easily analyzed, so assume $|\Lambda| \geq 2$.

Let $p \in \mathbb{H}^n$ be an arbitrary basepoint, and pick distinct points $x, y \in \Lambda$. Let $10\alpha > 0$ be the visual angle between x and y as seen from p.

Since Γp accumulates at x and y, we can find $g, h \in \Gamma$ with $d(p, g(p)), d(p, h(p)) > L(\alpha)$, and with the angle between $[p, g(p)]$ and $[p, h(p)]$ at least 9α.

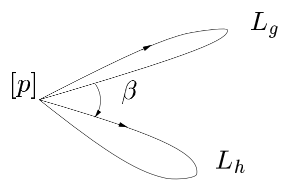

Figure 1.

We claim at least one of g, h and hg is hyperbolic. This is most easily seen in the quotient manifold. If the loop L_g to which $[p, g(p)]$ projects makes angle at least α at p, then it is represented by a geodesic, so g is hyperbolic. Similarly for L_h.

Otherwise, the angle β between any segment of L_g and any segment of L_h at p is at least $9\alpha - 2\alpha = 7\alpha$. Since $hg \in \pi_1(M, [p])$ is represented by $L_h * L_g$, and this composition lifts to a piecewise geodesic path in $\mathbb{H}^n$ making angles at least 7α at each vertex, the product hg is hyperbolic. ■

Another solution: one can quote:

Lemma 0.2 (Selberg) *Any finitely-generated subgroup Γ of $GL_n(\mathbb{R})$ contains a torsion-free subgroup of finite index.*

If Γ is a Kleinian group with all elements of finite order, then by Selberg's lemma the subgroup $\langle g, h\rangle$ generated by any $g, h \in \Gamma$ is finite. Thus the fixed-point sets of g and h intersect. Since the fixed-point sets are convex, all elements in Γ have a common fixed-point, and we conclude Γ is finite.

The idea of the proof of Selberg's lemma is to vary Γ slightly (specialize, in the sense of arithmetic geometry), so its generators are defined over a number field. Then only certain roots of unity can occur as eigenvalues of elements of Γ, so only finitely many different conjugacy classes of elements of finite order can occur. These conjugacy classes can be detected by reduction modulo a prime ideal, i.e. by a map of Γ to a finite group, and the kernel of this map is then torsion free.

For example, if $A \in SL_2(\mathbb{Z})$ is of finite order, its eigenvalues must be roots of unity that are quadratic over $\mathbb{Q}$, so A has order 2, 3, 4 or 6.

For details see [Rag, Cor 6.13]

Figure 2. Every closed path is a quasi-geodesic.

Show any non-elementary Kleinian group Γ contains a free group on two generators.

Solution. By the previous problem, we can find a hyperbolic element $g \in \Gamma$ stabilizing a geodesic γ_g. Replacing g with g^i if necessary, we can assume $\gamma_g/\langle g\rangle$ is a loop of length $L > L(90°)$.

Since Γ is nonelementary, $|\Lambda| > 2$. Using this fact, we can find a conjugate h of g such that its axis γ_h is disjoint from γ_g; indeed, we can arrange that $D = d(\gamma_h, \gamma_g) > L(90°)$ as well.

The desired free group is $\langle g, h\rangle \cong \mathbb{Z} * \mathbb{Z} \subset \Gamma$. Once again the result is most easily visualized in the quotient space $M' = \mathbb{H}^n/\langle g, h\rangle$. Consider abstractly two loops of length L joined by a segment of length D to form a barbell graph B. Map B into M' by sending the loops to the geodesic representatives of g and h, and sending the segment to their common perpendicular.

Any nontrivial element of $\pi_1(B) \cong \mathbb{Z} * \mathbb{Z}$ can be represented in M' be a piecewise geodesic loop making turns of $90°$ each at least distance $L(90°)$ apart. By the Lemma, each such loop is homotopic to a closed geodesic, so it is nontrivial in $\pi_1(M')$. Thus the group generated by g and h is free. ■

Riemann surfaces, dynamics and geometry

Harvard University — Spring 1998 — Math 275
C. McMullen

Selected Solutions

Prove that for any genus $g \geq 1$, the moduli space $\mathcal{M}_g$ has only one end. That is, any compact set $K' \subset \mathcal{M}_g$ is contained in a compact K with $\mathcal{M}_g - K$ connected.

The modular surface $\mathcal{M}_1$ is isomorphic to the $(2, 3, \infty)$-orbifold, so it clearly has one end (a neighborhood of the puncture).

Now suppose $g \geq 2$. Let $\mathcal{M}_g(\epsilon)$ denote the space of all surfaces whose shortest geodesic is of length less than $\epsilon > 0$. By Mumford's theorem $\mathcal{M}_g - \mathcal{M}_g(\epsilon)$ is compact, so it suffices to prove $\mathcal{M}_g(\epsilon)$ is connected.

So consider $X \in \mathcal{M}_g(\epsilon)$. We may assume ϵ is small enough that the geodesics Γ' on X of length shorter than ϵ are simple and disjoint. Enlarge Γ' to a *maximal* set of disjoint simple closed geodesics Γ. Then Γ dissects X into pairs of pants, so Γ serves to define Fenchel-Nielsen coordinates for $\mathcal{M}_g$ near X.

Let G be the trivalent graph with one vertex for each component of $X - \Gamma$ and one edge for each element of Γ, the edge joining the vertices for the components it borders. Let $Y(G)$ be the unique point in moduli space obtained by replacing each vertex of G by a pair of pants with cuffs of length $\epsilon/2$, and joining the pants without any twist. Since we can adjust the lengths of the curves in Γ at will, we can connect X to $Y(G)$ by a path remaining in $\mathcal{M}_g(\epsilon)$.

The graph G has $2g - 2$ vertices and $3g - 3$ edges, and the set of such graphs is finite. Moreover, any two such graphs are related by a sequence of Whitehead moves (exercise; see Figure 3).

When G and G' are related by a Whitehead move, the surfaces $Y(G)$ and $Y(G')$ can be joined by a path where the loops corresponding to the four exterior edges can remaing short. Thus $Y(G)$ and $Y(G')$ are in the same component of $\mathcal{M}_g(\epsilon)$. Thus any two points in $\mathcal{M}_g(\epsilon)$ can be connected by a path, and therefore $\mathcal{M}_g$ has only one end. ∎

Figure 3. Whitehead move.

References

[Rag] M. S. Raghunathan. *Discrete Subgroups of Lie Groups.* Springer-Verlag, 1972.