Riemann surfaces, dynamics and geometry

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6 Problems: Teichmüller theory and rational maps

1. Let $E \subset \widehat{\mathbb{C}}$ be a set of cardinality n, representing a point $X = \widehat{\mathbb{C}} - E \in T_{0,n}$ (for a suitable choice of marking).

   Discuss the isomorphism
   $$T_X T_{0,n} \cong T_E \widehat{\mathbb{C}} / sl_2(\mathbb{C}),$$
   describing the tangent space to Teichmüller space in terms of n-tuples of vectors
   $$T_E \widehat{\mathbb{C}} = \prod_{p \in E} T_p \widehat{\mathbb{C}},$$
   one at each point in E, modulo the subspace of n-tuples that extend from E to a holomorphic vector field $v \in sl_2(\mathbb{C})$ on $\widehat{\mathbb{C}}$. In particular, give a formula for the Teichmüller metric on this model of the tangent space.

   (Hint: for $\phi \in Q(X)$ and $v \in T_E \widehat{\mathbb{C}}$, there is a natural pairing $\langle v, \phi \rangle = \sum_E \mathrm{Res}_p(v\phi)$, where $v\phi$ is a 1-form with at worst a simple pole at p. Begin by proving that $\langle v, \phi \rangle = 0$ for $v \in sl_2(\mathbb{R})$.)

2. Let $\gamma \subset S$ be a closed loop and for each $X \in \mathrm{Teich}(S)$ let $\ell_\gamma(X)$ denote the length of the geodesic representative on γ on X (or zero if none exists). The function $\ell_\gamma : \mathrm{Teich}(S) \to \mathbb{R}$ is smooth, so $d\ell_\gamma$ is a 1-form on $\mathrm{Teich}(S)$, i.e. a section of the cotangent bundle of Teichmüller space.

   Find
   $$d\ell_\gamma(X) \in T^*_X \,\mathrm{Teich}(S) \cong Q(X)$$
   explicitly, i.e. compute $d\ell_\gamma(X)$ as a holomorphic quadratic differential on X.

3. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be a torus, and let $F : T^2 \to T^2$ be a covering map. When is F combinatorially equivalent to a holomorphic map on a complex torus?

4. Prove that for any critically finite branched covering $f : S^2 \to S^2$, we have $\sup \|DT_f\| = 1$ where the sup is over all points in $\mathrm{Teich}(S^2, P(f))$.

5. Let $F : S^2 \to S^2$ be a critically finite branched covering of degree two that looks like a quadratic polynomial which a *periodic* critical point. That is, assume the two branch points b_0 and b_∞ of F satisfy $F(b_\infty) = b_\infty$ and $F^n(b_0) = b_0$ for some n.

   Prove such an F is always combinatorially equivalent to a quadratic polynomial.