Riemann surfaces, dynamics and geometry

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5 Problems: Teichmüller geodesics and the mapping class group

1. Let $\phi \in Q(X)$ be a nonzero quadratic differential. Verify that the surfaces X_t with complex structure $t\overline{\phi}/|\phi|$ define a holomorphic, isometric map of the unit disk Δ to $\mathrm{Teich}(S)$ (using the metric of curvature -4 on the disk).

2. Show there is a natural action of $SL_2(\mathbb{R})$ on the cotangent space of Teichmüller space, such that the projection of the orbit of $\phi \neq 0$ is a complex Teichmüller geodesic.

   Intersection number and translation length. Let S be a compact oriented surface. For any two isotopy classes of closed curves α, β on S, the *geometric intersection number* $i(\alpha, \beta)$ is the minimum possible number of intersections of transverse representatives of α and β.

   A mapping class $\phi \in \mathrm{Mod}(S)$ is *reducible* if it preserves a nontrival system of disjoint simple-closed curves C on S up to isotopy. It is *pseudo-Anosov* if $i(\alpha, \phi^n(\beta)) \to \infty$ for any pair of simple closed curves α, β on S.

   The *translation length* $L(\phi)$ is defined by

   $$L(\phi) = \inf_{X \in \mathrm{Teich}(S)} d_T(X, \phi(X)).$$

3. Let S be a torus. Show, using Teichmüller theory, that the mapping class group of S is isomorphic to $SL_2(\mathbb{Z})$.

   (Hint: construct an exact sequence

   $$0 \to K \to \mathrm{Mod}(S) \to SL_2(\mathbb{Z}) \to 0,$$

   show K acts trivially on $\mathrm{Teich}(S)$ and interpret this fact geometrically.)

4. Let $X \in \mathrm{Teich}(S)$ be a torus endowed with a flat metric. Show for any simple closed curves α, β on S, the minimum intersection number $i(\alpha, \beta)$ is attained when α and β are representated by geodesics on X.

5. Now assume $\chi(S) < 0$. Pick any $X \in \mathrm{Teich}(S)$. Show $i(\alpha, \beta)$ is attained when α and β are representated by geodesics in the hyperbolic metric on X.

6. Let S be a torus. Show any $\phi \in \mathrm{Mod}(S)$ is either finite order, reducible, or pseudo-Anosov. Show this classification agrees with the (elliptic, parabolic, hyperbolic) classification of elements of $SL_2(\mathbb{Z})$.

7. Show that every finite order element $\phi \in SL_2(\mathbb{Z})$ can be realized as an automorphism of a complex torus.

8. Show that for every pseudo-Anosov $\phi \in SL_2(\mathbb{Z})$, there exists a $\lambda > 1$ such that

$$\lim_{n\to\infty} i(\alpha, \phi^n(\beta))^{1/n} = \lambda$$

for any pair of simple closed curves α and β on S. Relate λ to the translation length $L(\phi)$ and and to the trace of ϕ as a 2×2 matrix.

The classification of surface diffeomorphisms. For the remaining problems assume $\chi(S) < 0$. These problems lead through a proof that every $\phi \in \mathrm{Mod}(S)$ is either finite order, reducible or pseudo-Anosov.

9. Show if $\phi \in \mathrm{Mod}(S)$ has a fixed-point in $\mathrm{Teich}(S)$, then ϕ has finite order.

10. Show that if $L(\phi) = 0$ but ϕ has no fixed-point, then ϕ is reducible.

    (Hint: consider a sequence $X_n \in \mathrm{Teich}(S)$ such that $d_T(X_n, \phi(X_n)) \to 0$, and show $[X_n]$ must go to infinity in the moduli space $\mathcal{M}(S)$. Therefore the length of the shortest geodesic on X_n goes to zero. Use the short geodesics to construct an invariant curve system.)

11. More generally, show if $L(\phi)$ is not realized on $\mathrm{Teich}(S)$ (i.e. $d_T(X, \phi(X)) > L(\phi)$ for all X), then ϕ is reducible.

12. Finally suppose

$$d_T(X, \phi(X)) = L(\phi) > 0$$

for some $X \in \mathrm{Teich}(S)$. Let $f : X \to X$ be the Teichmüller mapping in the isotopy class of ϕ. Show that the dilatation of f satisfies $K(f)^2 = K(f \circ f)$.

(Hint: if $f \circ f$ is not a Teichmüller map, then $d_T(X, \phi^2(X)) < 2d_T(X, \phi(X))$, and one concludes $d_T(Y, \phi(Y)) < d(X, \phi(X))$ where Y is the midpoint of the geodesic segment from X to $\phi(X)$.)

Show that f preserves the foliations $\mathcal{F}(\phi)$ and $\mathcal{F}(-\phi)$, and stretches their leaves by constant factors in the $|\phi|$-metric.

From this prove that f is pseudo-Anosov. Conclude in fact that

$$\lim i(\alpha, \phi^n(\beta))^{1/n} = K(f) > 1$$

for any pair of simple closed curves α, β on S.