Riemann surfaces, dynamics and geometry

Harvard University — Spring 1998 — Math 275
C. McMullen

4 Problems: Teichmüller theory

1. Find a meromorphic quadratic differential ϕ on $\widehat{\mathbb{C}}$, with simple poles, such that
   - in the $|\phi|$-metric, the sphere is isometric to the double of a square, with vertices at $\{0, \infty, \pm 1\}$; and
   - $[0, 1]$ is a leaf of the foliation of ϕ.
2. Find a meromorphic ϕ on $\widehat{\mathbb{C}}$ such that in the $|\phi|$-metric, $\widehat{\mathbb{C}}$ is isometric to the double of an L-shaped polygon built from 3 squares.

   (Hint: find a suitable cubic polynomial $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and consider the pullback of the solution to (1)).
3. Let $\phi \neq 0$ be a quadratic differential on a finite area hyperbolic surface X, and let γ be a curve on X which is a geodesic for both the hyperbolic metric and the flat metric $|\phi|$. Prove that X is finitely covered by a real surface Y such that γ is contained in the projection of $Y(\mathbb{R})$. (A *real surface* (Y, C) is a Riemann surface equipped with an antiholomorphic involution C to play the role of complex conjugation; its real points $Y(\mathbb{R})$ are the fixed-points of C.)
4. Estimate the modulus of the Euclidean quadrilateral Q in $\mathbb{C}$ with vertices $\{0, T, T(1+i), i\}$, $T \gg 1$.
5. Estimate the modulus of the annulus $U_r = \Delta - [-r, r]$, with r close to 1.
6. Let $Q \subset \mathbb{C}$ be a quadrilateral, and let A, B be the minimum distances between the two pairs of opposite sides. Show $\text{area}(Q) \geq AB$.
7. Let Γ be a finitely generated Kleinian group with domain of discontinuity Ω. Prove that every component of Ω has nontrivial stablizer in Γ. (Hint: if not, then Γ has a wandering disk.)
8. Prove that for any genus $g \geq 1$, the moduli space $\mathcal{M}_g$ has only one end. That is, any compact set $K' \subset \mathcal{M}_g$ is contained in a compact K with $\mathcal{M}_g - K$ connected.
9. Let S be a surface of genus g. Show that $\text{Mod}(S)$ acts faithfully on $\text{Teich}(S)$ for $g \geq 3$, but not for $g = 1$ or $g = 2$.