Riemann surfaces, dynamics and geometry

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3 Problems

1. Show the post-critical set satisfies $P(f) = P(f^n)$ for any $n \geq 1$, and $|P(f)| \leq 2$ iff $f(z)$ is conjugate to $z \mapsto z^d$.

2. Prove that f is expanding iff there are no critical points or parabolic cycles in the Julia set $J(f)$.

3. Let T be an isoceles right triangle. Folding T along a line bisecting the right-angle, you obtain another triangle similar to T. Now the double of T is conformally isomorphic to the Riemann sphere; find an explicit degree two rational map corresponding to this folding operation.

4. Show the group $\operatorname{Aut}(f)$ of Möbius transformations commuting with a rational map f $(\deg(f) \geq 2)$ is finite.

5. Let $f(z) = z^2 + i$. Describe the dynamics of f on its Fatou set.

6. Suppose every critical point of f is preperiodic, but none is periodic. Show $J(f) = \widehat{\mathbb{C}}$.

7. Let $T_p(z)$ be Newton's method applied to a polynomial p. Show that the points of inflection (where $p''(z) = 0$) are the worst initial guesses. That is, if Newton's method works for every point of inflection, then it works for almost any other initial guess in $\mathbb{C}$.

8. Let $f(z) = 1/z$ for $|z| < 1$ and $f(z) = 0$ elsewhere. Show
$$\|f(z+t) - f(z)\|_1 = O(|t||\log t|)$$
as $t \to 0$ in $\mathbb{C}$. Conclude that any quasiconformal vector field v on $\mathbb{C}$ is continuous, and in fact locally
$$|v(z) - v(z+t)| = O(|t||\log t|),$$
so v is almost Lipschitz.

9. Let $U \subset \mathbb{C}$ be a bounded, simply-connected region. Show that the hyperbolic metric $\rho_U(z)\,|dz|$ satisfies
$$\rho_U(z) \asymp \frac{1}{d(z, \partial U)}.$$
Here $d(z, \partial U) = \inf\{|z - w| \ : \ w \in \partial U\}$, and $A \asymp B$ means $A/C < B < CA$ for some universal constant $C > 0$. (Hint: use the Schwarz lemma to get one bound, and the Koebe 1/4-theorem to get the other.)

10. Let $E \subset \mathbb{C}$ be a finite set of 2 or more points. Show that near each $w \in E$, the hyperbolic metric on the complement of E satisfies

$$\frac{|dz|}{|z-w|\,|\log|a(z-w)||} \;\le\; \rho_{\mathbb{C}-E}(z)\,|dz| \;\le\; \frac{|dz|}{|z-w|\,|\log|b(z-w)||},$$

for some $a, b > 0$ and all z close to w. In particular

$$\rho_{\mathbb{C}-E}(z)\,|dz| \sim \frac{|dz|}{|z-w|\,|\log|(z-w)||}.$$

(Hint: one direction is easy by the Schwarz Lemma. For the other, write $\widehat{\mathbb{C}} - E = \mathbb{H}/\Gamma$, and note that the covering map $\mathbb{H} \to (\mathbb{C} - E)$ factors through $\mathbb{H} \to \mathbb{H}/\Gamma_w \cong \Delta^*$, where $\Gamma_w \cong \mathbb{Z}$ is a parabolic subgroup of Γ corresponding to a small loop around w. The map $\Delta^* \to (\mathbb{C} - E)$ extends analytically to Δ, sending 0 to w, and we know the hyperbolic metric on Δ^*.)