Riemann surfaces, dynamics and geometry

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2 Problems: examples of rational maps

1. Let X be a closed Riemann surface of genus $g \geq 2$. Show any holomorphic map $f : X \to X$ is either constant or of finite order.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d > 1$, and let

$$\Omega_\infty(f) = \{z \ : \ f^n(z) \to \infty\}.$$

   Show $\Omega_\infty(f)$ is open.

   Show the Julia set $J(f)$ is equal to $\partial\Omega_\infty(f)$ and $J(f)$ is bounded.

   Show every bounded component of $\widehat{\mathbb{C}} - J(f)$ is a topological disk. (Hint: use the maximum principle and the normal families definition of $J(f)$.)

3. Show any degree two holomorphic map

$$q : \mathbb{C} \to \mathbb{C}$$

   is conjugate to a unique map of the form $f_c(z) = z^2 + c$.

   Show any pair (q, z) consisting of a quadratic polynomial and one of its fixed-points $z \in \mathbb{C}$ is conjugate to a unique pair of the form $(g_\lambda(z), 0)$, where $g_\lambda(z) = \lambda z + z^2$.

   Show that the forgetful map sending $(q, z) \to q$ induces the map

$$c(\lambda) = \frac{\lambda(2-\lambda)}{4}$$

   on the level of these canonical representatives. Explain why $c(\lambda)$ is branched over $c = 1/4$.

4. Let $C = \{c \in \mathbb{C} \ : \ f_c(z) = z^2 + c \ \text{ has an indifferent fixed-point}\}$. Show C is a cardioid; that is, C is similar to the image of the circle $|z - 1| = 1$ under $z \mapsto z^2$.

5. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and let $f : S^1 \to S^1$ be the map $f(x) = 2x \bmod 1$.

   Find all the periodic and preperiodic points of f. Show almost every $x \in S^1$ has a dense forward orbit in S^1. Show every $x \in S^1$ has a dense inverse orbit in S^1.

6. Show for every $\theta \in S^1$, there is an $x \in S^1$ and an ordering-preserving bijection sending $n\theta$ to $2^n x$. (In other words, f has an orbit with the same order structure as any rotation of the circle. For example, $\theta = 1/3$ corresponds to $x = 1/7$, which generates the cycle of period 3, $\langle 1/7, 2/7/4/7\rangle$.)

7. Let $B : \Delta \to \Delta$ be a proper holomorphic map of degree $d > 1$ of the unit disk into itself. Show B is a *Blaschke product* of the form

$$B(z) = \prod_1^d \frac{z - a_i}{1 - \overline{a}_i z}$$

with $a_i \in \Delta$. (Hint: consider $B^{-1}(0)$.)

8. Show B has a unique point fixed-point $p \in \overline{\Delta}$ such that $B^n(z) \to p$ for all $z \in \Delta$.

9. Considering $B(z)$ as a rational map, show $J(B) \subset S^1$. Show either $J(B) = S^1$ or $J(B)$ is a Cantor set.

10. Let f be a rational map. Show the forward orbit of a generic point in $J(f)$ is dense in $J(f)$. Here generic is in the sense of Baire category; i.e. show the set of such z is a dense G_δ (countable intersection of open sets).

11. For $t \in \mathbb{H}$ let $C = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}t)$ and let $\wp : C \to \widehat{\mathbb{C}}$ be the Weierstrass $\wp$-function. Find the rational function $f_t(z)$ such that

$$f_t(\wp(z)) = \wp(2z).$$

(Hint: C has an affine equation of the form $y^2 = 4x^3 + ax + b$; if $z = (x, y)$ lies in C, then $\wp(z) = x$ and $2z$ has a nice interpretation in terms of the tangent line to C at z.)

Show $J(f_t) = \widehat{\mathbb{C}}$, and f_s and f_t are topologically conjugate for any $s, t \in \mathbb{H}$. When are f_s and f_t conformally conjugate?

12. For $t \in \mathbb{H}$ let $X = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}t)$ and let $F : X \to X$ be given by $F(x) = nx$, $n > 1$. Show F is ergodic; that is, any measurable set $E \subset X$ with $F^{-1}(E) = E$ has zero or full measure.

(Hint: consider a point of Lebesgue density of E. Alternatively, reduce to the case $\tau = i$ and consider the Fourier series for $\chi_E(x + iy)$.)

Conclude that the rational maps $f_t(z)$ of the previous exercise are also ergodic.

13. Given a polynomial $p(z)$, *Newton's method* is the rational map

$$f_p(z) = z - \frac{p(z)}{p'(z)}.$$

Show if p is quadratic, then $f_p(z)$ is conjugate to z^2. Conclude that Newton's method is reliable for solving quadratics ($f_p^n(z)$ converges to a root of p for almost any initial guess z). Describe the locus of bad initial guesses geometrically.