Riemann surfaces, dynamics and geometry

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1 Problems: examples of hyperbolic manifolds

1. Compute the hyperbolic metric on $A(r) = \{z \,:\, r < |z| < 1\}$.
2. Show each conjugacy class in the group $\mathrm{Aut}(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C})$ has a representative of the form $g(z) = z + 1$ or $g(z) = \lambda z$, $\lambda \in \mathbb{C}^*$. Show the representative is unique up to replacing λ by λ^{-1}.
3. Find the centralizers of $g(z) = z + 1$ and $g(z) = \lambda z$ in $\mathrm{Aut}(\widehat{\mathbb{C}})$.
4. Show that if $X = \mathbb{H}/\Gamma$ then $\mathrm{Aut}(X) = N(\Gamma)/\Gamma$, where $\mathrm{Aut}(X)$ is the group of holomorphic automorphisms of X and $N(\Gamma)$ is the normalizer of Γ in $\mathrm{Aut}(\mathbb{H})$.
5. Find all Riemann surfaces X for which $\mathrm{Aut}(X)$ is an uncountable group (equivalently, $\mathrm{Aut}(X)$ contains a 1-parameter subgroup).
6. Show any infinite Kleinian group Γ contains an element of infinite order. (Hint: if $H \subset \mathbb{H}^n$ is a half-space, and $\gamma(H)$ is a proper subset of H, then γ has infinite order.)
7. A discrete group is *elementary* if it contains an abelian subgroup with finite index.

   Show any Kleinian group $\Gamma \subset \mathrm{Aut}(\widehat{\mathbb{C}})$ with nontrivial center is elementary.
8. Show any non-elementary Kleinian group Γ contains a free group on two generators.
9. Let $W \subset S^3$ be the Whitehead link and let $B \subset S^3$ be the Borromean rings.

   Show $S^3 - W$ and $S^3 - B$ are commensurable; that is, there is a single 3-manifold M which is a finite cover of both.

   Using the fact that both link complements are hyperbolic, conclude that their hyperbolic volumes satisfy $\mathrm{vol}(S^3 - B) = 2\,\mathrm{vol}(S^3 - W)$.
10. Let $M = [0,1]^3 - I_a - I_b$, where I_a and I_b are two disjoint cylinders joining a pair of horizontal sides of the cube and a pair of vertical sides, respectively (Figure 1). Observe that M is a 3-manifold with boundary, homeomorphic to a handlebody of genus two.

    Show that $\pi_1(M) \cong \langle a, b\rangle$ is a free group generated by loops a and b around the cylinders I_a and I_b.

    Show that the commutator $[a,b] \in \pi_1(M)$ can be represented by a simple closed curve on ∂M, indeed by a loop running along the edges of the cube and visiting every vertex exactly once (a *Hamiltonian cycle*).

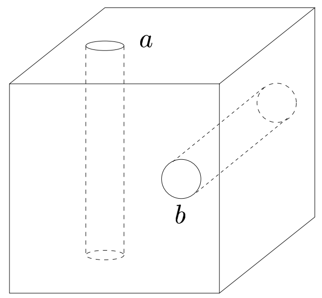

Figure 1. A handlebody cut from a cube.

11. Let B be a Klein bottle, i.e. a closed nonorientable surface with $\chi(B) = 0$. Show that the interior of $B \times [0,1]$ admits a complete hyperbolic structure.

12. Now let M be an orientable 3-manifold fibering over B with fiber $[0,1]$. (Such a manifold is called an I-*bundle* over B.) Show $\text{int}(M)$ admits a complete Euclidean structure, but does *not* admit a complete hyperbolic structure.