Conformal Dynamics Homework 6
Math 275 — Harvard University — Fall 2001

1. Let $X = \mathbb{C}/\Lambda$ be a complex torus. Given an example of a smooth, locally affine map $f : X \to X$ which is *not* a Teichmüller map.

   (This means (a) near every point $p \in X$ there is a complex coordinate system in domain and range such that $f(x + iy) = x + Kiy$; but (b) one cannot find a holomorphic quadratic differential $\phi \in Q(X)$ such that $\mu(f) = k\overline{\phi}/|\phi|$.)

2. Let $f(z)$ be a critically finite rational map. Prove that the Julia set of f is connected.

3. Prove that $\chi(\mathcal{O}_f) \leq 0$ for any critically finite branched covering $f : S^2 \to S^2$.

4. Give an example of a critically finite branched cover of the sphere with orbifold of signature $(2, 2, 2, 2)$ that is *not* combinatorially equivalent to a rational map.

5. Let $f : S^2 \to S^2$ be a critically finite branched covering such that every branch point is periodic, and such that $f^{-1}(p) = p$ for some $p \in P(f)$. Prove that f is combinatorially equivalent to a polynomial.

6. Prove that the boundary of the Mandelbrot set is not a countable union of smooth arcs.

7. Let $\Gamma \cong \mathbb{Z} * \mathbb{Z}$ be a Kleinian group with connected domain of discontinuity $\Omega \subset \widehat{\mathbb{C}}$. Suppose that $X = \Omega/\Gamma$ is a closed surface of genus two. Prove that the limit set $\Lambda(\Gamma)$ is a Cantor set.

8. Prove that $\Lambda(\Gamma)$ has measure zero.

9. Give a detailed proof of the puzzle theory result that 'an only child is excellent'; include an analysis of the cases not treated by Milnor.

10. Let $s(z) = z^2$ and let $E \subset S^1$ be a closed set such that $s(E) = E$ and $s|E$ is a bijection. Prove that E is finite.