Conformal Dynamics Homework 4
Math 275 — Harvard University — Fall 2001

1. Let $X = \mathbb{H}/\Gamma$ where Γ is generated by $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. Prove that X is isomorphic to the hexagonal torus $\mathbb{C}/\mathbb{Z} \oplus \omega\mathbb{Z}$ with a point removed (here $\omega^3 = 1$).

2. Let S be the punctured torus S with $\pi_1(S) = \langle a, b\rangle$. Is the word $ab^2ab^3a^2b$ represented by a simple closed curve on S?

3. Prove that the punctured square torus can be represented as $\mathbb{H}/\langle A, B\rangle$, where $\operatorname{tr} A = \operatorname{tr} B = 2\sqrt{2}$. What is $\operatorname{tr} AB$? Show that A and B can be chosen to lie inside the image of $\mathrm{GL}_2(\mathbb{Q})$ in $\mathrm{PGL}_2(\mathbb{R})$, but not inside the image of $\mathrm{GL}_2(\mathbb{Z})$.

4. Show that the group of all outer automorphisms of $G = \mathbb{Z} * \mathbb{Z}$ is generated by three 'reflections' (i.e. elements of order 2). Describe the natural action of these reflections on (a) the space of simple closed curves on the punctured torus, (b) on $\mathbb{H}$, and (c) on the representation variety $\mathcal{V}(G)$.

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1. Let $\phi : A(R) \to A(R)$ be a quasiconformal automorphism of an annulus, commuting with the action of an irrational rotation. Prove that ϕ is Lipschitz.

2. Let $f(z)$ be a polynomial of degree d, and define $h : \mathbb{C} \to \mathbb{R}$ by
$$h(z) = \lim_{n\to\infty} |f^n(z)|^{1/d^n}.$$
Show that the limit above exists, that $h(f(z)) = h(z)^d$, that $U = \{z : h(z) > 0\}$ coincides with the attracting basin of infinity, and that $\partial U = J(f)$.

   Show that for $z, w \in U$, we have $h(z) = h(w)$ iff the closures of the small orbits of z and w agree. Thus h gives the foliated small orbit quotient on the basin of infinity.

3. Consider the family of cubic polynomials
$$f_t(z) = -z^3 + 12z + t$$
over $X = \{t : |t| < 1\}$.

   Show that f_t is J-stable but not topologically stable. Sketch a picture of the topologically stable regime. What is happening dynamically at the topologically unstable points?