Conformal Dynamics Homework 3

Math 275 — Harvard University — Fall 2001

1. Let $U \subset \widehat{\mathbb{C}}$ be the image of the unit disk under a K-quasiconformal map. Let $f : \Delta \to U$ be a Riemann mapping. Prove a bound of the form $\|Sf\| = O((K-1)/(K+1))$.

2. Let $\rho_t : G \to \operatorname{Aut}\widehat{\mathbb{C}}$ be a holomorphic family of representations over a connected base X, and suppose $\rho_t(G)$ is discrete and nonelementary for all t. Prove that all the representations in the family are quasiconformally conjugate.

3. Give a counterexample where $\rho_t(G)$ is allowed to be elementary.

4. Let $\Gamma \subset \operatorname{Aut}\widehat{\mathbb{C}}$ be a finitely-generated group in which every element has finite order. Prove that Γ is finite.

5. Let $G = \langle g, h\rangle \subset \operatorname{Aut}\widehat{\mathbb{C}}$ be a solvable nonabelian group other than $\mathbb{Z}/2 \ltimes \mathbb{Z}$. Show that g and h have a common fixed-point.

6. Prove that every I-bundle over a closed surface S (orientable or not) with $\chi(S) < 0$ carries a convex cocompact hyperbolic structure.

7. Let $\gamma_i \subset X_i$, $i = 1, 2$, be a pair of simple closed geodesics on a pair of closed hyperbolic surfaces. Let $h : \gamma_1 \to \gamma_2$ be a diffeomorphism, and let $Y = X_1 \cup_h X_2$ be the 2-complex obtained by gluing γ_1 to γ_2 with h. Finally let Γ be a Kleinian group isomorphic to $\pi_1(Y)$.

   Show that the limit set of Γ is connected.

8. Construct a convex cocompact Kleinian group isomorphic to $\pi_1(Y)$.

9. Given an example of a quasifuchsian group Γ and a $z \in \mathbb{H}^3$ such that $\overline{D(z)} \cap S^2_\infty$ has more than 2 components, where $D(z)$ is the Dirichlet domain for z. (Hint: let $\Gamma = Q(X, \tau^n(Y))$ where τ is a Dehn twist.)

Let X be a compact Riemann surface of genus 2 with hyperelliptic involution $f : X \to X$.

1. Show that $f^*\phi = \phi$ for every holomorphic quadratic differential on X.

2. Let $p \in X$ be any point. Show that $\dim Q(X - \{p\}) = 1 + \dim Q(X)$.

3. Let $p \in X$ be a fixed-point of f (a Weierstrass point). Show that there exists a nonzero $\psi \in Q(X - \{p\})$ such that $f^*\psi = -\psi$.

4. Let $P : Q(X - \{p\}) \to Q(X)$ be the unique projection with $P(\psi) = 0$. Prove directly that $\|P\| = 1$ with respect to the Teichmüller (L^1) norm. (Hint: use the fact that $|a+b| + |a-b| \geq 2|a|$.)

5. Give another proof that $\|P\| = 1$ using the fact that the Weierstrass points provide sections of the universal curve, $s : \mathcal{T}_2 \to \mathcal{C}_2$. (Hint: interpret P as $(Ds)^*$.)

6. Now let $p \in Y$ be a point on a compact Riemann surface of genus $g \geq 3$. Prove there is no projection
$$P : Q(Y - \{p\}) \to Q(Y)$$
with $\|P\| = 1$. Conclude there is no section of the universal curve $\mathcal{C}_g \to \mathcal{T}_g$.