Conformal Dynamics Homework 2
Math 275 — Harvard University — Fall 2001

1. Let M be the Mandelbrot set. Show that the set of $c \in M$ such that $J(f_c)$ is expanding accumulates at every point on ∂M.

2. Show that for a 'generic' point $c \in \partial M$ (in the sense of Baire category), the map f_c has no indifferent cycle.

3. Show that if f is an expanding rational map, then the Julia set of f is quasi-self-similar. That is, given $K > 1$, there exists an $r > 0$ such that for any $x \in J(f)$ and any ball B centered at x of radius $< r$, there is a K-quasi-similarity $\phi : B \to B'$ such that $\operatorname{diam} B' > r$ and $\phi(J \cap B) = J \cap B'$. (A K-quasi-similarity distorts ratios of distances by at most a factor of K.)

4. Show that if $J \subset \mathbb{C}$ is a compact, nowhere dense, quasi-self-similar set, then $\operatorname{H.dim} J < 2$.

5. Suppose $J(f_c)$ has positive measure. Show that $d(f^n(z), P(f)) \to 0$ for almost every $z \in J(f)$, where $P(f)$ is the closure of the forward orbit of the critical point $z = 0$.

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1. Prove $\|Sf\| \leq 3/2$ for univalent maps on $\mathbb{H}$ using the area theorem.

2. Let $\mu \in M(\widehat{\mathbb{C}})$ be an L^∞ Beltrami differential. Compute a formula for $df_t(z)/dt$, where $f_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the unique quasiconformal map fixing 0, 1 and ∞ with $\mu(f_t) = t\mu$.

3. Show that for any two Riemann surfaces $X, Y \in \operatorname{Teich}(S)$, with $\delta = d_T(X,Y) \leq \log\sqrt{2}$, there exists a unique quasiconformal map $f : X \to Y$ respecting markings with $\mu(f)$ a harmonic Beltrami differential. Give an upper bound for $K(f)$ in terms of δ.

4. Suppose there exists a $c_0 \in C$ such that $f_c(z) = z^2 + c$ carries an invariant linefield on its Julia set. Prove that c_0 is a stable parameter for the family f_c, and that expanding dynamics is not dense in the Mandelbrot set.

5. Prove that a Kleinian group $\Gamma \subset \operatorname{Aut}\widehat{\mathbb{C}}$ is convex cocompact iff it is expanding. The latter condition means for all $x \in \Lambda(\Gamma)$ there exists a $\gamma \in \Gamma$ with $|\gamma'(x)| > 1$ in the spherical metric.

6. Let $Q(R) = [0,1] \times [0,R] \subset \mathbb{C}$. Show there is a constant $c > 0$ such that $\|Sf\| > c$ for any $\epsilon > 0$ and any conformal map $f : Q(1) \to Q(1+\epsilon)$. Here the norm of the Schwarzian is given by $\|Sf\| = \sup \rho^{-2}_{Q(1)}|Sf|$.

7. Let $E \subset \mathbb{C}$ be a set consisting of n points. Let $V \cong \mathbb{C}^E$ be the vector space of sections of $T\mathbb{C}$ over E.

   Describe the natural pairing between V and $Q(\widehat{\mathbb{C}} - E)$, obtained by letting $v \in V$ represent a tangent vector to the Teichmüller space $T_{0,n}$ at E, and by letting $\phi \in Q(\widehat{\mathbb{C}} - E)$ represent a cotangent vector. Which elements of V pair trivially with every $\phi \in Q(\widehat{\mathbb{C}} - E)$?