Conformal Dynamics Homework 1
Math 275 — Harvard University — Fall 2001

1. Classify Möbius transformations $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ up to topological conjugacy. Which ones are structurally stable?

2. Classify projective automorphisms $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ up to topological conjugacy. Which ones are structurally stable? (Note: f belongs to $PGL_3(\mathbb{R})$.)

3. Classify expanding maps on the n-torus $X = \mathbb{R}^n/\mathbb{Z}^n$ up to topological conjugacy. (Here an endomorphism $f : X \to X$ is *expanding* if for some $\lambda > 1$ we have $\|Df(v)\| \geq \lambda\|v\|$ for every $v \in TX$.

A *continuous motion* of $X \subset Y$ is a continuous map $f : [0,1] \times X \to Y$ such that $f_t(x)$ is a injective for each fixed $t \in [0,1]$ and $f_0(x) = x$.

1. Give an example of a finite 1-complex $X \subset S^2$ and a homeomorphism $f : X \to Y \subset S^2$ that does not extend to a homeomorphism of S^2 to itself.

2. Show that any continuous motion of a closed set $X \subset S^1$ extends to a continuous motion of the whole circle.

3. Show that any continuous motion of a finite set $X \subset S^2$ extends to a continuous motion of the whole sphere.

4. Given an example of a continuous motion of a countable closed set $X \subset S^2$ that does not extend to a continuous motion of the whole sphere.

A *holomorphic motion* of $X \subset \widehat{\mathbb{C}}$ over a Riemann surface (T, p) is a map $f : T \times X \to \widehat{\mathbb{C}}$ such that $f_t(x)$ is holomorphic in t for each fixed x, injective in x for each fixed t, and $f_p(x) = x$.

1. Consider the holomorphic motion of $X = \mathbb{Z} \oplus \mathbb{Z}i \subset \mathbb{C}$ over $(\mathbb{H}, i)$ given by $f_t(n + mi) = n + tm$. Show that f_t extends to a holomorphic motion of $\mathbb{C}$.

2. Show that the extension of f in the preceding question is *unique*. In fact, for any $z \notin X$, the extension of f to a holomorphic motion of $X \cup \{z\}$ is unique.

3. Let $\Gamma \subset \mathrm{Aut}(\Delta)$ be a torsion-free discrete group of Möbius transformations acting on the disk. Show that $f_t(\gamma(0)) = \gamma(t)$ defines a holomorphic motion of $X = \Gamma \cdot 0$ over $(\Delta, 0)$.

4. Show that if Δ/Γ is isomorphic to $\widehat{\mathbb{C}} - \{0, 1, \infty\}$, then the holomorphic motion f just defined extends to a holomorphic motion of the whole disk.

5. Show that this extension is unique, and that $t \mapsto f_t(x)$ is *not* a Möbius transformation if $t \notin \Gamma \cdot 0$.

6. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a K-quasiconformal map. Show that for any 4-tuple of distinct points on $\widehat{\mathbb{C}}$, the cross-ratio satisfies

$$d([a, b, c, d], [f(a), f(b), f(c), f(d)]) \leq \log K,$$

where $d(\cdot)$ denotes the hyperbolic metric on $\widehat{\mathbb{C}} - \{0, 1, \infty\}$.

7. Show conversely that if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is an orientation-preserving homeomorphism with the cross-ratio property above (for some K), then f is quasiconformal.

8. Compute the Ahlfors-Weill extension for the univalent map $f(z) = z^\alpha$ on $\mathbb{H}$, α close to 1. For what range of α's do we obtain a quasiconformal extension?