Complex Dynamics and Hyperbolic Geometry Problems

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1 Basic properties

1. Let Γ be the infinite cyclic group of homeomorphisms of $\mathbb{C}^* = \mathbb{C} - \{0\}$ generated by $T(x + iy) = x/2 + 2iy$. Show Γ has discrete orbits, but its action on $\mathbb{C}^*$ is not properly discontinuous. That is, exhibit a compact set $K \subset \mathbb{C}^*$ such that $T^i(K) \cap K \neq \emptyset$ for infinitely many i.

2. Let $G = \mathrm{Aut}(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C})$ be the group of Möbius transformations. Let $A \subset G$ be a discrete subgroup isomorphic to $\mathbb{Z}^2$. Show that A is conjugate in G to a lattice of translations generated by $z \mapsto z + 1$ and $z \mapsto z + \tau$, for some $\tau \in \mathbb{H}$.

   What happens if we allow A to be indiscrete?

3. Show that $z(s) = \tanh(s) + i \operatorname{sech}(s)$ gives the arclength parameterization of a geodesic in $\mathbb{H}$.

2 One-dimensional hyperbolic space

1. For any interval $I = (a, b) \subset \mathbb{R} \subset \mathbb{RP}^1$, define the *hyperbolic metric* on I by $\rho_I = 2|dx|/(b - x)(x - a)$.

   Show that ρ is invariant under the stabilizer of I in the group $PGL_2(\mathbb{R})$ of projective automorphisms of $\mathbb{RP}^1$.

2. For a map $f : I \to J$ between intervals in $\mathbb{RP}^1$, the *Schwarzian derivative* is the quadratic differential given by

$$Sf(x) = \left(\left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 \right) dx^2.$$

   Show that $Sf > 0$ implies f is distance-decreasing between the hyperbolic metrics on I and J.

3. Given a convex set $\Omega \subset \mathbb{RP}^n$, we may similarly define the *Hilbert metric* ρ_Ω to be the largest metric such that every projective embedding of an interval $f : I \to \Omega$ is distance-non-increasing from ρ_I to ρ_Ω. By definition, the Hilbert metric is invariant under projective automorphisms of Ω.

   Show that any projective map $f : \Omega \to \Omega'$ between convex sets is distance-decreasing for the Hilbert metric.

4. Use the preceding contraction principle to prove the Perron-Frobenius theorem (on uniqueness of the positive eigenvalue of a positive matrix).

5. Here is a complex version of the Hilbert metric. For any domain $\Omega \subset \mathbb{CP}^1$, the *projective metric* is the largest metric on Ω such that every complex projective map $f : \mathbb{H}^2 \to \Omega$ is distance-decreasing.

   Describe the projective metric on the complement of 3 points; on the complement of 4 points; and on the region $\Omega = \{z \,:\, 0 < \arg z < \theta\}$.

6. Show the projective metric on a domain $\Omega \subset \mathbb{CP}^1$ always has curvature $K \le 0$.

3 Examples of hyperbolic manifolds

1. For $R > 1$ let $A(R) = \{z \in \mathbb{C} \;:\; 1 < |z| < R\}$. Find a subgroup $\Gamma \subset \operatorname{Isom} \mathbb{H}$ such that $A(R)$ is isomorphic to $\mathbb{H}/\Gamma$ as a Riemann surface. Compute the (holomorphic) universal covering map $\pi : \mathbb{H} \to A(R)$ explicitly.

2. Let

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;:\; a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

   and let $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$. Show the region

$$F = \{z \in \mathbb{H} \;:\; |\operatorname{Re} z| \le 1/2 \text{ and } |z| \ge 1\}$$

   is a fundamental domain for $PSL_2(\mathbb{Z})$.

   (Hint: The group $SL_2(\mathbb{Z})$ acts transtively on the set of oriented bases of $\mathbb{Z}^2$. Given a lattice $\Lambda \subset \mathbb{C}$ with $z = 1$ as a shortest vector, a shortest vector in $\Lambda - \mathbb{Z}$ lies in F.)

3. Describe the quotient $\mathbb{H}/PSL_2(\mathbb{Z})$. Using your description, show that $PSL_2(\mathbb{Z})$ is isomorphic to the free product $(\mathbb{Z}/2) * (\mathbb{Z}/3)$.

4. Show that $\mathrm{PSL}_2(\mathbb{Z})$ has elements of orders 1, 2 and 3 but no higher order.

5. Let

$$\Gamma(n) = \{A \in SL_2(\mathbb{Z}) \;:\; A = I \bmod n\}/\{\pm I\} \subset PSL_2(\mathbb{Z}).$$

   Show that $\Gamma(2)$ is torsion-free.

6. Show that $\mathbb{H}/\Gamma(2)$ is a triply-punctured sphere, and thus $\Gamma(2)$ is isomorphic to the free product $\mathbb{Z} * \mathbb{Z}$.

7. Let $X(n) = \mathbb{H}/\Gamma(n)$ as a Riemann surface. By the preceding exercise, $X(2)$ is isomorphic to $\widehat{\mathbb{C}} - \{0, 1, \infty\}$. Give a similar explicit description of $X(n)$ for $n = 3, 4, 5, 6$.

8. For any group G, let G' be its commutator subgroup, i.e. the group generated by $[a, b] = aba^{-1}b^{-1}$ for all $a, b \in G$.

   Let $G = PSL_2(\mathbb{Z})$. Show $G/G' \cong \mathbb{Z}/6$, and $G'/G'' \cong \mathbb{Z}^2$.

   Describe the Riemann surfaces $\mathbb{H}/G'$ and $\mathbb{H}/G''$ explicitly.

9. (Hard) Show that the complement of the Whitehead link is homeomorphic to $\mathbb{H}^3/\Gamma$ for a subgroup of finite index $\Gamma \subset PSL_2(\mathbb{Z}[i])$. (Hint: one can take a regular ideal octahedron in $\mathbb{H}^3$ as a fundamental domain for Γ.)

4 Ergodic theory

1. Let $\Sigma_n = (\mathbb{Z}/n)^{\mathbb{Z}}$ endowed with the natural invariant probability measure. Show that the shift map $\sigma : \Sigma_n \to \Sigma_n$ is mixing. (Hint: regard σ as an automorphism of a compact topological group.)

2. Use harmonic extension from the circle S^1 to the unit disk $\Delta \subset \mathbb{C}$ to show any irrational rotation $T(z) = e^{i\theta} z$ on S^1 is ergodic.

3. Consider a proper holomorphic map $T : \Delta \to \Delta$ of degree $d > 1$ with $T(0) = 0$. Using harmonic functions, show that the boundary map $T : S^1 \to S^1$ is measure-preserving, ergodic and mixing.

   (Hint: for mixing, use the fact that $T^n(z) \to 0$ for all $z \in \Delta$, by the Schwarz lemma.)

4. Let $f : S^1 \to S^1$ be a smooth *expanding* map: that is, $\inf |f'| > \lambda > 1$.

   (a) Prove that f is ergodic. (Note: f need not be measure-preserving; here *ergodic* means any f-invariant set has measure 0 or 1.)

   (b) Prove there is a homeomorphism $h : S^1 \to S^1$ conjugating $f(z)$ to $s(z) = z^d$ for some d, $|d| > 1$.

   (c) Prove that either $dh/d\theta = 0$ a.e. or h is C^∞. (Here $z = \exp(i\theta)$.)

5. Let $U : H \to H$ be a unitary operator, with spectral decomposition $H = \int_{S^1} H_\lambda \, dm$. What is the spectral decomposition of U^2?

6. Let $T : \mathbb{Z} \to \mathbb{Z}$ be the shift map, $T(n) = n + 1$, and let U be the corresponding unitary operator on $\ell^2(\mathbb{Z})$.

Show that U has has Lebesgue spectrum of multiplicity one. That is, in the spectral decomposition

$$\ell^2(\mathbb{Z}) \cong \int_{S^1} H_\lambda \, dm(\lambda),$$

we can take $dm = d\theta$ and $\dim H_\lambda = 1$.

7. (Open problem.) Give an example of an ergodic automorphism of a *finite* measure space whose unitary operator has Lebesgue spectrum of finite multiplicity.

8. Show that any two hyperbolic automorphisms T_1, T_2 of the torus $G = \mathbb{R}^2/\mathbb{Z}^2$ determine conjugate unitary automorphisms U_1, U_2 of $L^2(G)$.

9. More generally, show that for any ergodic automorphism T of a compact abelian group G, the corresponding unitary operator U on $L^2_0(G)$ has Lebesgue spectrum of infinite multiplicity.

10. Prove that if $T : X \to X$ is ergodic and has Lebesgue spectrum on $L^2_0(X)$, then T is mixing.

5 Hyperbolic geometry

1. Let $\mathbb{R}^{2,1}$ denote $\mathbb{R}^3$ with the Minskowski metric $x^2 + y^2 - t^2$. Show the space $\mathcal{G}$ of oriented geodesics in $\mathbb{H}$ can be identified with the unit sphere in $\mathbb{R}^{2,1}$.

2. Show that for any $v, w \in \mathcal{G} \subset \mathbb{R}^{2,1}$, we have

$$\langle v, w \rangle = \cos(\theta)$$

if the corresponding geodesics in $\mathbb{H}$ meet in angle θ.

What is the geometric interpretation of $\langle v, w \rangle$ when the geodesics do not meet? In particular, what does its sign mean?

3. Find R and r such that the circles of radius r centered at $(\pm R, \pm iR)$ bound a regular hyperbolic quadrilateral in the unit disk Δ with interior angles of $45°$.

4. Describe the invariant Lorentz metric on $\mathcal{G}$. When is the distance between a pair of points in $\mathcal{G}$ positive (spacelike)? When is it negative (timelike)?

5. Describe the invariant measure on $\mathcal{G}$. Show that, suitably normalized, the geodesics meeting a segment $L \subset \mathbb{H}$ have measure $\ell(L)$. Show the geodesics meeting a compact convex set $K \subset \mathbb{H}$ ($\operatorname{int} K \neq \emptyset$) have measure $\ell(\partial K)/2$. (Here $\ell(\cdot)$ denotes hyperbolic length.)

6. Consider the following dictionary:

Torus $X = \mathbb{R}^2/\mathbb{Z}^2$	Unit tangent bundle $X = T^1 S$
Hyperbolic automorphism $T \in SL_2\mathbb{Z}$	Geodesic flow g^t on X
Translation h^s along $\mathcal{F}^u$	Horocycle flow h^s
$\langle T, h^s \rangle$ generate a solvable group	$\langle g^t, h^s \rangle$ generate a solvable group
T and h^s are ergodic	g^t and h^s are ergodic
T is mixing	g^t is mixing
h^s is not mixing	h^s is mixing

Explain conceptually why the last line is different. (One approach: use the fact that there are two horocycle flows in both cases.)

7. Using the geometry of hyperbolic 1-manifolds, estimate the number of integer solutions to $x^2 = 1 + 2y^2$ satisfying $|x| < T$. Relate your answer to the group of units in the ring of integers $\mathcal{O} \subset \mathbb{Q}[\sqrt{2}]$. (Hint: consider the norm map $N : \mathcal{O} \to \mathbb{Z}$.)

8. Let $q(x, y) = x^2 - Dy^2$. Describe the map

$$X = SO(q, \mathbb{R})/SO(q, \mathbb{Z}) \to SL_2(\mathbb{R})/SL_2(\mathbb{Z})$$

for $D = 7$ and for $D = 9$. In particular, describe geometrically the projection of X to the modular surface $\mathbb{H}/SL_2(\mathbb{Z})$.

9. Estimate the number of points in the orbit $SL_2(\mathbb{Z}) \cdot i \subset \mathbb{H}$ satisfying $d(x, i) \leq 10$. Then check your answer by a direct calculation.

6 Quasiconformal maps

1. Construct a quasiconformal homeomorphism $f : [0, 1] \to [0, 1]$ whose distributional derivative $f'(x)$ is not in $L^1[0, 1]$; equivalently, such that $f(E)$ has positive measure for some closed set $E \subset [0, 1]$ of measure zero.

2. Construct a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ which is analytic outside a closed set of measure zero, but not globally analytic.

3. Show $f(x, y) = (g(x), y)$ is a quasiconformal homeomorphsim iff $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz homeomorphism.

4. Prove directly from the definition that a geometrically quasiconformal mapping $f : S^n \to S^n$ is Hölder continuous; that is, $d(f(x), f(y)) = O(d(x, y)^{1/K})$ for some K.

5. Show directly that the vector field $v = \mu * (1/z)$ has an $|x \log x|$ modulus of continuity, for any $\mu \in M(\Delta)$, where $\Delta = \{z \in \mathbb{C} \ : \ |z| < 1\}$. (Hint: show that $\int_\Delta |\phi| = O(|\log p|)$ for $\phi = dz^2/(z(z - p))$.)

7 Ergodic theory of Kleinian groups

1. Let $M_1(\widehat{\mathbb{C}})$ be the unit ball in the Banach space of L^∞ Beltrami differentials on $\widehat{\mathbb{C}}$. Define $F : \mathrm{Aut}(\widehat{\mathbb{C}}) \times M_1(\widehat{\mathbb{C}}) \to M_1(\widehat{\mathbb{C}})$ by $F(g, \mu) = g^* \mu$. Show that F is continuous if we use the weak* topology on $M_1(\widehat{\mathbb{C}})$, but discontinuous if we use the norm topology.

2. Using promotion, show the action of $\pi_1(M)$ on S^n_∞ is ergodic for any finite-volume hyperbolic manifold.

3. Let X be a compact hyperbolic surface of genus two, let $f : X \to \partial H$ map X to the boundary of a handlebody of genus two, so the map

$$f_* : \pi_1(X) \to \pi_1(H) \cong \mathbb{Z} * \mathbb{Z}$$

is surjective. Let $Y \to X$ be the covering space of X corresponding to $\mathrm{Ker}\, f_*$. Show that the limit set of $\pi_1(Y) \subset \mathrm{Isom}\, \mathbb{H}^2$ is the whole circle S^1_∞, but the action of $\pi_1(Y)$ on S^1_∞ is not ergodic.

4. Let Γ be a Kleinian group, let $x \in \mathbb{H}^3$ be any point, and let

$$F = \{y \in \mathbb{H}^3 \ : \ d(y, x) = \inf_{\Gamma} d(y, \gamma(x))\}.$$

Show that F is a fundamental domain for Γ acting on $\mathbb{H}^3$.

Now let $E = \overline{F} \cap \Lambda$. Show that the action of Γ on Λ is conservative iff $m(E) = 0$. More precisely, show that Γ acts conservatively on $\Lambda - \Gamma \cdot E$.

8 Ahlfors' finitness theorem

1. Prove that if X has infinite genus, then $\dim Q(X) = \infty$. (Let (a_i, b_i) be an infinite collection of disjoint simple curves on X with $a_i \cdot b_i = 1$. Construct square-integrable holomorphic 1-forms θ_i with $\int_{a_j} \theta_i = \delta_{ij}$. Then show $\langle \theta_i^2 \rangle$ are linearly independent in $Q(X)$.

2. Prove that if X has finite genus but infinite area, then $\dim Q(X) = \infty$. (Let $D_i = B(x_i, r_i) \subset X_i$ be a sequence of embedded hyperbolic disks with $r_i \to \infty$. Let $\pi_i : \Delta \to X$ be the universal covering map with $\pi_i(0) = x_i$. Let $\phi_j = z^j\, dz^2 \in Q(\Delta)$. Show that the quadratic differentials $(\pi_i)_*(\phi_j) \in Q(X)$ are linearly independent for $j = 0, 1, 2, \ldots, N_i$ with $N_i \to \infty$.)