**Final Solutions**
Real Analysis
Math 212a – Harvard University – Fall 1998

1. Let $f : [0,1] \to \mathbb{R}$ be differentiable; that is, assume $f'(x)$ exists at each $x$. Show that $f$ is Lipschitz on some open interval $(a,b) \subset [0,1]$.

   Proof. The continuous functions $f_t(x) = t^{-1}(f(x+t) - f(x))$ converge to $f'(x)$ pointwise, so $M(x) = \sup_t |f_t(x)|$ is finite for each $x$. By the uniform boundedness principle, there is an open set $U$ on which $M(x)$ is uniformly bounded, and then $f|U$ is Lipschitz because $|f'|$ is bounded.

2. Let $H \subset C[0,1]$ denote the set of increasing homeomorphisms $h : [0,1] \to [0,1]$. Prove that a generic $h \in \overline{H}$ (in the sense of Baire category) is a homeomorphism and satisfies $h'(x) = 0$ a.e.

   Proof. Given any rationals $x \neq y$ in $[0,1]$, the set of $h \in \overline{H}$ such that $h(x) \neq h(y)$ is open and dense, so a generic $h$ is a homeomorphism.

   Setting $f_t(x) = t^{-1}(f(x+t) - f(x))$, let $U_t \subset \overline{H}$ be the open set of $f$ such that $|f_t| < t$ on a set of measure $> 1 - t$. Any homeomorphism $h$ can be approximated by an $f \in U_t$ with many small flat steps, so $U_t$ is open and dense. Taking the intersection over $t = 2^{-n}$ we conclude that $h'(x) = 0$ a.e. for a generic $h \in \overline{H}$.

3. Prove there is a closed set $F \subset [0,1] \times [0,1]$ such that for any closed $E \subset [0,1]$, there exists a $t$ such that $E = F_t = \{x \ : \ (x,t) \in F\}$.

   Proof. Enumerate the open sets in $[0,1]$ by $U_1, U_2, \ldots$ and let $K$ be the standard Cantor set. Define $F \subset [0,1] \times K$ by writing $t \in K$ as $t = 0.t_1 t_2 \ldots$ in base 3, and then setting $F_t = [0,1] - \bigcup_{t_i = 0} U_i$. Since the set where $t_i = 0$ is open in $K$, $F$ is closed, and clearly $E = F_t$ where the digits of $t$ are 0 or 2 according to whether $U_i \cap E = \emptyset$ or not.

4. We say $f_n \to f$ *monotonely* on $[0,1]$ if $f_n \to f$ pointwise and $f_1 \leq f_2 \leq \cdots$ or $f_1 \geq f_2 \geq \cdots$. A function $f$ is of *Baire class* 0 if $f$ is continuous, and of Baire class $n+1$ if it is a monotone limit of functions of Baire class $n$.

   Show any bounded measurable function $g : [0,1] \to \mathbb{R}$ agrees with a function $f$ of Baire class 2 outside a set of measure zero.

   Proof. Baire class 1 includes indicator functions of closed sets, so Baire class 2 includes indicator functions of $F_\sigma$'s. Since every measurable set agrees with an $F_\sigma$ a.e., the theorem is true when $g = \chi_E$. A similar argument is holds for step functions, and then for limits of step functions, which give all $g$.

5. Show the same statement is false for $f$ of Baire class 1.

   Proof. Let $E \subset [1/2, 1]$ be a Cantor set of positive measure, and let $g(x) = \chi_E(x) - \chi_E(x + 1/2)$. Then $g$ cannot agree a.e. with a function $f$ of Baire class 1, because $f$ is either upper or lower semicontinuous.

6. Prove that Tychonoff's Theorem implies the Axiom of Choice.

   Proof: The Axiom of Choice says for any collection of nonempty sets $\{X_\alpha\}$, the product $X = \prod X_\alpha$ is nonempty. To prove this, choose an auxiliary point $p$, let $Y_\alpha = X_\alpha \sqcup \{p\}$, give $Y_\alpha$

the topology with open sets $\{\emptyset, \{p\}, X_\alpha, Y_\alpha\}$. Then $Y_\alpha$ is compact, so by Tychonoff's Theorem $Y = \prod Y_\alpha$ is also compact.

Let $E_\alpha = \{y \in Y \,:\, y_\alpha \in X_\alpha\}$. Then $\langle E_\alpha \rangle$ is a family of closed sets with the finite intersection property, since for any finite set of indices we can choose elements $x_\alpha \in X_\alpha$. By compactness, $X = \bigcap E_\alpha \neq \emptyset$.

7. Let $\Delta = \{z \in \mathbb{C} \,:\, |z| < 1\}$, and let $\mathcal{F}$ be the family of all holomorphic functions $f : \Delta \to \overline{\Delta}$. Show that $\mathcal{F}$ is compact in the compact-open topology.

   Proof: Since $\mathcal{F}$ is equicontinuous and $\Delta$ is locally compact, the compact-open topology is the same as the topology of pointwise convergence. Since $\mathcal{F}$ forms a normal family, any sequence has a convergent subsequence, and thus $\mathcal{F}$ is compact.

8. Let $H^2(\Delta)$ be the Hilbert space of holomorphic functions $f(z)$ on $\Delta$ such that

$$\|f\|_2^2 = \int_\Delta |f(z)|^2 \, |dz|^2 < \infty.$$

   Show that for $z \in \Delta$, $\phi_z(f) = f(z)$ is a bounded linear functional. Find $K_z \in H^2(\Delta)$ such that $\phi_z(f) = \langle f, K_z \rangle$.

   Answer: By the mean-value theorem, $f(0)$ is the average of $f$ over the disk, so we can take $K_0(z) = \pi^{-1}$. By naturality with respect to Möbius transformations, we then find $K_z(w) = 1/(\pi(1-\overline{z}w)^2)$.

9. Let $B = \{a \,:\, \|a\|_p \leq 1\} \subset \ell^p(\mathbb{N})$ be the unit ball, endowed with the weak topology. Prove that for $1 < p < \infty$, the functions of the form $f(a) = q(a_0, a_1, \ldots, a_n)$, where $q$ is a polynomial in $a_0, \ldots, a_n$, are dense in $C(B)$.

   Proof. For $1 < p < \infty$ the space $\ell^p$ is reflexive, so the weak and weak* topologies on $B$ coincide, and thus $B$ is a compact Hausdorff space by Alaoglu's theorem. The polynomials above form a subalgebra of $C(B)$ that separates points, and by the Stone-Weierstrass theorem this algebra is dense in $C(B)$.

10. Show the same result fails for $p = \infty$.

    Proof. Let $\phi \in C(B)$ be a bounded linear functional such that $\phi(a) = \lim a_n$ when $a_n$ converges. Given a polynomial $q(a_0, a_1, \ldots, a_n)$, construct a point $a \in B$ by setting $a_i = 0$ for $i \leq n$ and $a_i = \pm 1$ for $i > n$, with sign chosen to disagree with the sign of $q(0, \ldots, 0)$. Then $|q(a) - \phi(a)| \geq 1$, so $\phi$ is not in the closure of the polynomials in $C(B)$.

11. Let $A : X \to Y$ be a linear map between Banach spaces that is continuous from the weak topology on $X$ to the norm topology on $Y$. Show that $A(X)$ is finite-dimensional.

    Proof. Let $B$ be the unit ball in $Y$. Then $A^{-1}(B)$ contains a basic weak neighborhood of zero, which is defined by the inequalities $|\phi_i(x)| < 1$ for a finite set $\phi_1, \ldots, \phi_n \in X^*$. Thus the finite codimension subspace $Z = \bigcap \operatorname{Ker} \phi_i$ is contained in $A^{-1}(B)$. By homogeneity, $Z$ is contained in $\operatorname{Ker} A$, so $A(X) \cong X/Z$ is finite dimensional.

12. Give an example of a convex set $K$ in a Banach space $X$, such that $0 \notin K$, but every $\phi \in X^*$ has a zero in $K$. Show that $K$ cannot be closed.

Answer: An example is provided by the set $K \subset \mathbb{R}^2$ consisting of those $(x, y)$ with $y > 0$ or $y = 0$ and $x > 0$. If $K$ were closed then it would be separated from the origin by a linear functional $\phi$, by the separation theorem; but then $\phi$ would have no zero on $K$.

13. Let $X$ be a normal topological space. State and prove a theorem characterizing those sets $Z \subset X$ which can arise as zero sets of continuous functions $f : X \to \mathbb{R}$.

Answer: $Z$ is a zero-set iff $Z$ is a closed $\mathcal{G}_\delta$. Proof: $Z = \bigcap f^{-1}(-1/n, 1/n)$, so $Z$ is a closed $\mathcal{G}_\delta$. Conversely, if $Z$ is a closed $G_\delta$ then we can write $Z - X = \bigcup F_n$ where $F_n$ are closed. By Urysohn's lemma, there is a continuous $f_n : Z \to [0, 1]$ with $f_n = 0$ on $Z$ and $f_n = 1$ on $F_n$. Then $Z$ is the zero set of $f(x) = \sum 2^{-n} f(x)$.

14. Suppose $f \in C[0, 1]$ satisfies $\int_0^1 g''(x) f(x)\, dx \leq \|g\|_\infty$ for all smooth functions $g \in C^\infty[0, 1]$. Show that $f = k - h$ where $k$ and $h$ are convex.

Proof. This formula means that the distribution $f''$ is a signed measure, say $f'' = \mu - \nu$, where $\mu$ and $\nu$ are positive measures. Then the second integrals $k$ and $h$ of $\mu$ and $\nu$ are convex functions, and (for the right constants of integration) we have $f = k - h$.

15. Let $K(X)$ be the collection of all nonempty compact subsets of a compact metric space $(X, d)$. For $K_1, K_2 \in K(X)$ and $r > 0$ let $B(K_i, r)$ denote an $r$-neighborhood of $K_i$ in $X$ and define

$$d(K_1, K_2) = \inf\{r > 0 \ : \ K_1 \subset B(K_2, r) \ \&\ K_2 \subset B(K_1, r)\}.$$

Show that $(K(X), d)$ is a compact metric space.

Proof: if $K_i$ is a Cauchy sequence in $K(X)$ then it is not hard to see that $K_i \to L$ where $L = \bigcap_n \overline{\bigcup_{i>n} K_i}$. Thus $K(X)$ is complete. Given $r$, we can choose a finite set $E \subset X$ with $B(E, r) = X$; then every compact $K \subset X$ is within distance $r$ of a subset of $E$, and thus $K(X)$ is totally bounded. Therefore $K$ is compact.

16. Compute the Hausdorff dimension of $(K[0, 1], d)$. (Cf. Royden, Ch. 12 problem 55.)

Answer: the Hausdorff dimension is infinite, since $n$-pointed sets give copies of $[0, 1]^n$ inside $K[0, 1]$.