SOLUTION. We can find the kernel but not the image. The dimensions of the image and kernel are 2
(2 pivot, 2 nonpivot columns). The linear system to B is x + 2y + 5z = 0, z + 3s = 0. Solving gives
u = 1, z = -3t, y = s, x = -2a - 5t, so that a general kernel element is [t | -5, 0, -3, 1] or [-2 | 1, 0, 0].

PROBLEM Given the basis \{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \} in the plane. a) Find the coordinates of \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} in this basis. b) Find a vector with coordinates \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} in that basis, find the coordinates in the standard basis.

SOLUTION. S = \begin{bmatrix} 4 & 7 & 2 \\ 1 & 2 & 7 \end{bmatrix} has the inverse \begin{bmatrix} 2 & 7 & -4 \\ -1 & 2 & 7 \end{bmatrix}. a) The vector has new coordinates \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}. b) The vector has the standard coordinates \begin{bmatrix} 10 \\ 3 \end{bmatrix}.

PROBLEM Let A be a shear along a line L. Find ker(A - I), im(A - I) and \((A - I)^2\).

SOLUTION. Shears have the property that \((A - I)x = Ax - x = pL, therefore \im(A - I) = L\). The kernel of \(A - I\) consists of vectors \(Ax = x\). Because every \(v \in L\) has this property, the kernel is \(L\) too. Both have dimension 1. We have \((A - I)^2 = 0\) because the image of \(A - I\) is a subset of the kernel.

PROBLEM Let L be the line spanned by \(\vec{v} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}\) and let T be the counterclockwise rotation about an angle \(\pi/2\) around L (this means \(\pi/2\) clockwise if you look from \(\vec{v}\) to the origin). Find the matrix A.

SOLUTION. Draw a good picture. \(\vec{e}_1\) goes to \(\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}\) and \(\vec{e}_2\) goes to \(\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}\). These are the columns of A, so

\[ A = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \]

PROBLEM. Let A be a 3 \times 3 matrix satisfying \(A^2 = 0\). Show that the image of A is a subset of the kernel of A and determine all possible values for rank(A). Give an example.

SOLUTION. If x is in the image then \(x = Ay\) and \(Ax = A^2y = 0\) so that x is in the kernel of A. A is not invertible and cannot have rank 3. It can be the 0 matrix with rank 0. It can have rank 1: take 0 in the first two columns and \(\vec{e}_1\) in the last. It cannot have rank 2 because the dimension of the kernel would be 1 and could not contain the kernel.

PROBLEM. A is a rotation-dilation, a composition of a rotation with \(\pi/2\) and dilation by 4. Take B as a rotation-dilation with angle \(\pi/4\) and dilation factor 2. The rank of B is 2 because if it were smaller, then also the rank of A were smaller. \(B^{17}\) is a rotation dilation with angle \(17\pi/2 \sim \pi/2\) and dilation factor 217.

SOLUTION. Yes: the linear map which maps the span of \(\vec{e}_1\) into the span of \(\vec{w}_1\) is invertible: it has the matrix

\[ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \]

PROBLEM. Find the rank of the matrix A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 100 \\ 101 \\ 102 \\ 103 \\ ... \\ 100 \end{bmatrix}.

SOLUTION. Deleting the first row from each other row makes the lower 99 rows all linearly dependent. The rank is 2.