COMPUTING EIGENVALUES

Math 21b, O.Knill

THE TRACE. The trace of a matrix $A$ is the sum of its diagonal elements.

EXAMPLES. The trace of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ is $1 + 4 + 8 = 13$. The trace of a skew symmetric matrix $A$ is zero because there are zeros in the diagonal. The trace of $I_n$ is $n$.

CHARACTERISTIC POLYNOMIAL. The polynomial $f_A(\lambda) = \det(\lambda I_n - A)$ is called the characteristic polynomial of $A$.

EXAMPLE. The characteristic polynomial of $A$ above is $x^3 - 13x^2 + 5x$.

The Theorem. The trace of the characteristic polynomial $f_A(\lambda)$.

Proof. If $\lambda$ is an eigenvalue of $A$ with eigenvector $v$, then $A - \lambda I$ has $v$ in the kernel and $A - \lambda I$ is not invertible so that $f_A(\lambda) = \det(A - \lambda I) = 0$.

The polynomial has the form $f_A(\lambda) = \lambda^n - \text{tr}(A)\lambda^{n-1} + \cdots + (-1)^n\det(A)$.

THE 2x2 CASE. The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc)$.

The eigenvalues of $A$ are $\lambda_1 = \frac{a + d + \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$ and $\lambda_2 = \frac{a + d - \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$.

THE FIBONNACCI RABBITS. The Fibonacci's recursion $u_{n+1} = u_n + u_{n-1}$ defines the growth of the rabbit population. We have seen that it can be rewritten as $A^n = A^n u_1 = A^n u_0$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The roots of the characteristic polynomial $p_A(x) = x^2 - 1$ are $1, 1$.

ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_1)^k(\lambda - \lambda_2)^{n-k}$, then the eigenvalue is $\lambda_1$ with algebraic multiplicity $k$.

EXAMPLE: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2.

HOW TO COMPUTE EIGENVECTORS? Because $(\lambda - A)v = 0$, the vector $v$ is in the kernel of $\lambda - A$. We know how to compute the kernel.

EXAMPLE FIBONNACCI. The kernel of $M_2 - A = \begin{bmatrix} \lambda_1 - 1 & -1 \\ -1 & \lambda_2 \end{bmatrix}$ is spanned by $\mathbf{v}_\perp = \begin{bmatrix} 1 & -1 \end{bmatrix}$.

SOLUTION OF FIBONNACCI. To obtain a formula for $A^n\mathbf{v}$ with $f_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we form $f_A^n = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Let $\mathbf{v}_\perp = \begin{bmatrix} 1 & -1 \end{bmatrix}$, then $u_n = \begin{bmatrix} 11 & 1 \\ 1 & 0 \end{bmatrix}^n\mathbf{v}_\perp = (1 + \sqrt{5})^n/\sqrt{5}$.

ROOTS OF POLYNOMIALS. For polynomials of degree 3 and 4 there exist explicit formulas in terms of radicals. As Galois (1811-1832) and Abel (1802-1829) have shown, it is not possible for equations of degree 5 or higher. Still, one can compute the roots numerically.