

# TUTORIAL SPRING 2005: COMPLEX MULTIPLICATION

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## TENTATIVE SYLLABUS

- February 8: Introduction I – Galois Theory / Cyclotomic Fields (Teru)
- February 15: Introduction II – Elliptic Functions / Modular Functions (Teru)
- February 22: CM values of  $j(\tau)$  (Teru)
- March 1: Algebraic Curves (Sug Woo)
- March 8: Elliptic Curves over  $\mathbb{C}$  and  $\mathbb{F}_q$  (Sug Woo)
- March 15: Algebraic Proof of the First Main Theorem (Sug Woo)
- March 22: Adelic Formulation of the Main Theorems (Teru)
- March 29 Spring Break
- April 5/12/19/26, May 3: TBA

Before the spring break, one of our main goals would be to understand what's going on in the 5-page proof of the main theorems of complex multiplication by J.-P. Serre contained in the Brighton proceedings *Algebraic Number Theory* edited by Cassels and Fröhlich [4]. On the other hand, we intend to give a reasonable introduction on the origin of algebraic geometry - how elliptic functions and modular functions, which are meromorphic complex functions, evolved into elliptic curves and modular curves, which are the examples of algebraic curves (over  $\mathbb{C}$ , over  $\mathbb{Q}$ , and even over  $\mathbb{Z}$ !). Explicit computations on the special values of special analytic functions as  $j(\tau)$  and  $\wp(z)$  would be ultimately understood in terms of more intrinsic arithmetic algebraic objects. These lead to the great stream of arithmetic geometry of elliptic curves and modular curves, which for example lead to the solution of Fermat's Last Theorem by A. Wiles and R. Taylor in 1995 and is still being a very active area of research.

We will see where we are at the Spring Break, when you can spend some time to summarize what you'd have learned so far. And we might assign some materials to each of you and ask you to present at the meetings. There are many topics that we'd like to cover - but most importantly  $\ell$ -adic Galois representations attached to Hecke characters and elliptic curves, as it seems to be the most relevant formulation of the theory of complex multiplication in the modern terminology.

As the course goes through, the main obstacles regarding the necessary knowledge would be that of algebraic number theory (decomposition of primes and class field theory) and algebraic geometry. If you are eager to learn these topics, go ahead, and we can provide more challenging materials for you — if you aren't, just try to get

interested in these theories through this tutorial. After all, Complex Multiplication is undoubtedly one of the most beautiful example of these theories.

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