$K \supset \mathcal{O} \supset \mathfrak{p}$. $k = \mathcal{O}/\mathfrak{p} \cong \mathbb{F}_q$.

$k \subset \kappa \subset \overline{\mathbb{F}_q}$. $\mathcal{O}_\kappa :=$ complete unram. ext. of $\mathcal{O}$ w/ res. field κ.

Def. $\mathcal{C} :=$ (category of <u>Artinian</u> local $\mathcal{O}_\kappa$-alg. $(A, \mathfrak{m})$ s.t. $\kappa \xrightarrow{\sim} A/\mathfrak{m}$) [by str. mor. $\mathcal{O}_\kappa \to A$]

$\downarrow$ full [i.e. $\forall A, B \in \mathcal{C}$. $\mathrm{Hom}_{\mathcal{C}}(A,B) \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathcal{C}}}(A,B)$]

$\hat{\mathcal{C}} :=$ (category of <u>complete noetherian</u> —— " ——)

Morphisms : <u>local</u> $\mathcal{O}_\kappa$-alg. hom [i.e. $f : A \to A'$. $f(\mathfrak{m}) \subset \mathfrak{m}'$].

Rem. $\mathcal{C}$ has i) <u>final object</u> κ. $\hat{\mathcal{C}}$ has <u>initial object</u> $\mathcal{O}_\kappa$.

ii) <u>fibre product</u> $A \times_C B$

Recall : fibre prod. in (Sets): $f : X \to Z$. $g : Y \to Z \Rightarrow X \times_Z Y := \{(x,y) \in X \times Y \mid f(x) = g(y)\}$

in $\forall$ category:
$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ f\downarrow & \circlearrowright & \downarrow f' \\ Y & \longrightarrow & Y' \end{array}$$
$\Rightarrow \exists! h : X \longrightarrow Y \times_{Y'} X'$ s.t. $f = pr_1 \circ h$. $g = pr_2 \circ h$.

in (Sets): given by $h = (f, g) : x \longmapsto (f(x), g(x))$

this diagram is <u>Cartesian</u> if h : isom. (write $\square$)

in (Sets): (f, g) : bij. $\Leftrightarrow$ $Y \ni \forall y \longmapsto y' \in Y'$. $g : f^{-1}(y) \xrightarrow{\sim} f'^{-1}(y')$

[$\Rightarrow (y, x') \mapsto g^{-1}(x')$: inverse of h]

· <u>fibre prod</u> in $\mathcal{C}$ or (Rings) : component wise operation on fibre prod as sets.

$\angle$ <u>direct prod</u> = fibre prod / final obj. ⋯ $\times_\kappa$ in $\mathcal{C}$. $\neq$ direct prod as <u>rings</u>. (final obj in (Rings) ⋯ zero ring).

Def. A <u>deformation functor</u> on $\mathcal{C}$ is a covariant functor $F : \mathcal{C} \to (\mathrm{Sets})$ s.t. i) $F(\kappa) = \{\cdot\}$ ii) $F(A \times_C B) \xrightarrow{\sim} F(A) \times_{F(C)} F(B)$.

Rem. We can extend F to $\hat{F} : \hat{\mathcal{C}} \to (\mathrm{Sets})$ by $\hat{F}(R) := \varprojlim_n F(R/\mathfrak{m}^n)$ for $\forall (R, \mathfrak{m}) \in \hat{\mathcal{C}}$. [$R/\mathfrak{m}^n \in \mathcal{C}$]

Def. F : <u>formally smooth</u> $\Leftrightarrow F(A) \twoheadrightarrow F(A/I)$ ($\forall I \subset A \in \mathcal{C}$ ideal)

Ex.
- $R \in \hat{\mathcal{C}} \Rightarrow \mathrm{Spf}\, R : \mathcal{C} \ni A \longmapsto \mathrm{Hom}_{\hat{\mathcal{C}}}(R, A) \in (\mathrm{Sets})$: deform. funct. [<u>pro-represented</u> by R].
- $f : R \longrightarrow R'$ in $\hat{\mathcal{C}} \Rightarrow f^* : \mathrm{Spf}\, R' \longrightarrow \mathrm{Spf}\, R$: mor. of functors. ($\mathrm{Hom}(R', A) \ni \varphi \mapsto \varphi \circ f \in \mathrm{Hom}(R, A)$, "$f^*\varphi$")
- $\mathrm{Spf}\, \mathcal{O}_\kappa$: formally sm. [$\because$) $\mathrm{Hom}_{\hat{\mathcal{C}}}(\mathcal{O}_\kappa, A) = \{\cdot\}$ $\forall A$]
- $\mathrm{Spf}\, \kappa$: not fm. sm. [$\because$) $\underset{= \phi}{\mathrm{Hom}_{\mathcal{C}}(\kappa, \mathcal{O}_\kappa/\mathfrak{p}^2)} \longrightarrow \underset{= \{\cdot\}}{\mathrm{Hom}_{\mathcal{C}}(\kappa, \mathcal{O}_\kappa/\mathfrak{p})}$]
- $\mathrm{Spf}\, \mathcal{O}_\kappa[[X_1, \ldots, X_n]]$: fm. sm. [$\because$) $\mathrm{Hom}_{\hat{\mathcal{C}}}(\mathcal{O}_\kappa[[X_1, \ldots, X_n]], A) \xrightarrow{\sim} \mathfrak{m}^n$, $f \longmapsto (f(X_1), \ldots, f(X_n))$; $A \twoheadrightarrow A' \Rightarrow \mathfrak{m} \twoheadrightarrow \mathfrak{m}'$]

$\downarrow$

We'll see the "converse".

Def. $(\mathrm{Vect}_k) :=$ (category of fin. dim. vect. sp. $/k$)

$\mathcal{C}^\circ :=$ (category of $A \in \mathcal{C}$ s.t. $\begin{cases} A: k\text{-alg.} \\ m^2 = 0 \end{cases}$) $\xhookrightarrow{\text{full}} \mathcal{C}$

$m \in (\mathrm{Vect}_k)$

- $(\mathrm{Vect}_k) \xleftrightarrow[\sim]{\text{equiv. of cat.}} \mathcal{C}^\circ$

$m \longleftarrow A$

$V \longmapsto k[V] \; (= k[V]/(V^2))$

$k[V] := k \oplus V$ as k-vect. sp. ($\mathcal{O}_k$-mod.)
ring by $(a+v)(a'+v') = aa' + (av' + a'v)$

$\mathrm{Hom}_k(m, m') \cong \mathrm{Hom}_{\mathcal{C}^\circ}(A, A')$

in particular: i) $0 \longmapsto k$ (final & initial obj.)

ii) $V \oplus V' \longmapsto k[V] \times_k k[V']$ (direct ~~prod~~ & direct sum in $\mathcal{C}^\circ$)

iii) $\mathcal{C}^\circ$: additive category.

Lemma 1 F: deform. funct.
$\Rightarrow F|_{\mathcal{C}^\circ}$ factors thru an additive functor to (Vect_k)

$$\begin{tikzcd} (\mathrm{Vect}_k) \simeq \mathcal{C}^\circ \arrow[r, hook] \arrow[d, dashed] & \mathcal{C} \arrow[d, "F"] \\ (\mathrm{Vect}_k) \arrow[r, "\text{forget}"'] & (\mathrm{Sets}) \end{tikzcd}$$

$\because)$ k-vect. sp. $=$ "k-vect. sp. obj." in (Sets).

i.e. obj. X + mor. $\begin{cases} X \times X \xrightarrow{+} X, & 0: (\text{final}) \to X \\ X \xrightarrow{a} X & (\forall a \in k) \end{cases}$

satisfying:

$$\begin{tikzcd} & X \times X \arrow[dr, "+"] & \\ X \times X \times X \arrow[ur, "+\times id"] \arrow[dr, "id \times +"'] & \circlearrowleft & X \\ & X \times X \arrow[ur, "+"'] & \end{tikzcd} \qquad \begin{tikzcd} X \times X \arrow[r, "+"] \arrow[d, "a \times a"'] & X \arrow[d, "a"] \\ X \times X \arrow[r, "+"] & X \end{tikzcd} \quad \text{etc.}$$

$+$ on Hom.

$$\begin{tikzcd} X \arrow[r, "\Delta"'] \arrow[rrr, bend left, "f+g"] & X \times X \arrow[r, "f \times g"'] & Y \times Y \arrow[r, "+"] & Y \end{tikzcd}$$

$\Delta = (id, id)$

$\forall A \in \mathcal{C}^\circ$: k-vect. sp. obj. in $\mathcal{C}$.
$\Longrightarrow F(A) \in (\mathrm{Vect}_k)$ [F commutes w/ fibre prod.]

Def. · $TF: (\mathrm{Vect}_k) \simeq \mathcal{C}^\circ \xrightarrow{F|_{\mathcal{C}^\circ}} (\mathrm{Vect}_k)$ [i.e. $TF(V) := F(k[V])$]

· $\phi: F \longrightarrow G \Rightarrow T\phi: TF \longrightarrow TG$ [i.e. $T\phi(V) := \phi(k[V])$].

Ex.

1) $T\,\mathrm{Spf}\, k[V] = \mathrm{Hom}_k(V, -)$ [$\because)$ $T\,\mathrm{Spf}\,k[V](V') = \mathrm{Hom}_{\mathcal{C}}(k[V], k[V']) = \mathrm{Hom}_k(V, V')$]

2). $(R, m_R) \in \hat{\mathcal{C}}$. R: k-alg. $T_R := m_R/m_R^2$

$f: R \twoheadrightarrow k[T_R]$. $T\,\mathrm{Spf}\,R(V) = \mathrm{Hom}_{\hat{\mathcal{C}}}(R, k[V]) \xleftarrow[f^*]{\sim} \mathrm{Hom}_{\mathcal{C}}(k[T_R], k[V]) = T\,\mathrm{Spf}\,k[T_R](V)$

$\Rightarrow Tf^*: T\,\mathrm{Spf}\,k[T_R] \xrightarrow{\sim} T\,\mathrm{Spf}\,R$.

3) $\forall R \in \hat{\mathcal{C}}$. $f: R \twoheadrightarrow R \otimes_{\mathcal{O}_k} k$. $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, k[V]) \xleftarrow[f^*]{\sim} \mathrm{Hom}_{\hat{\mathcal{C}}}(R \otimes k, k[V])$

$\Rightarrow Tf^*: T\,\mathrm{Spf}(R \otimes k) \xrightarrow{\sim} T\,\mathrm{Spf}\,R$.

ex. $f: \mathcal{O}_k[[X_1, \cdots, X_n]] \twoheadrightarrow k[X]$. when $X := \bigoplus_{i=1}^n k \cdot X_i \in (\mathrm{Vect}_k)$.

$\Rightarrow Tf^*: \mathrm{Hom}_k(X, -) = T\,\mathrm{Spf}\,k[X] \xrightarrow{\sim} T\,\mathrm{Spf}\,\mathcal{O}_k[[X_1, \cdots, X_n]]$.

[$f: R \twoheadrightarrow k[T_{R \otimes k}]$. $\Rightarrow Tf^*: \mathrm{Hom}_k(T_{R\otimes k}, -) \cong T\,\mathrm{Spf}\,R$]

Lemma 2. $I \subset A \in \mathcal{C}$. $\mathfrak{m}I = 0$ $[\Rightarrow I \in (\mathrm{Vect}_k)]$

Then:

$$a : A \times_k k[I] \longrightarrow A, \quad (x, \bar{x}+t) \longmapsto x+t$$

gives

$$\begin{tikzcd} A \times k[I] \arrow[r, "a"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\square"] & A \arrow[d] \\ A \arrow[r] & A/I \end{tikzcd}$$

[in particular. F: deform. funct.

$\Downarrow$

$$\begin{tikzcd} F(A) \times TF(I) \arrow[r, "+", "F(a)"'] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\square"] & F(A) \arrow[d, "f"] \\ F(A) \arrow[r, "f"] & F(A/I) \end{tikzcd}$$

i.e. $\forall x \in F(A) \longmapsto y \in F(A/I)$

$$TF(I) \xrightarrow[\text{bij.}]{\sim} f^{-1}(y), \quad t \longmapsto x+t \quad]$$

∵) $(pr_1, a) : A \times_k k[I] \longrightarrow A \times_{A/I} A$

$$(x, \bar{x}+t) \longmapsto (x, x+t)$$

$\cdots$ bij. $\mathcal{O}_K$-lin. ring hom. $\square$

Lemma 3. i) $T\phi : TF \xrightarrow{\sim} TG \iff T\phi(k) : TF(k) \xrightarrow{\sim} TG(k)$.

ii) If F: formally sm. $\phi : F \xrightarrow{\sim} G \iff T\phi : TF \xrightarrow{\sim} TG$.

∵) i): true for $\forall$additive funct on (Vect_k). take $k^n \xrightarrow{\sim} \forall V \in$.

$$\begin{tikzcd} TF(k)^n \arrow[r, phantom, "\cong"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & TF(k^n) \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & TF(V) \arrow[d] \\ TG(k)^n \arrow[r, phantom, "\cong"] & TG(k^n) \arrow[r, "\sim"] & TG(V) \end{tikzcd}$$

ii). $\Rightarrow$: triv. $\Leftarrow$: take $I \subset A \in \mathcal{C}$. $\mathfrak{m}I = 0$.

We have

$$\begin{tikzcd} F(A) \arrow[r, "\phi(A)"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & G(A) \arrow[d, "f'"] \\ F(A/I) \arrow[r, "\phi(A/I)"'] & G(A/I) \end{tikzcd}$$

$\cdots\cdots$ ∵) $F(A/I) \ni \forall y \longmapsto y' \in G(A/I)$. take $x \in F(A)$, $x \mapsto y$.

$$\begin{tikzcd} TF(I) \ni t \arrow[r, "\sim", "\text{lem. 2}"'] \arrow[d, "T\phi(I)\cong"', "\text{assump.}"] & x+t \in f^{-1}(y) \arrow[d, "\cong : \text{claim}"] \\ TG(I) \ni t' \arrow[r, "\sim"', "\text{lem. 2}"] & x'+t' \in f'^{-1}(y') \end{tikzcd}$$

commutes

$$\left[\begin{tikzcd} F(A) \times TF(I) \arrow[r, "+"] \arrow[d, "\phi(A)"'] \arrow[d, "T\phi(I)"] \arrow[dr, phantom, "\circlearrowright"] & F(A) \arrow[d, "\phi(A)"] \\ G(A) \times TG(I) \arrow[r, "+"] & G(A) \end{tikzcd}\right]$$

$\Downarrow$

$\phi(A/I)$: bij $\Longrightarrow \phi(A)$: bij.

$A \twoheadrightarrow \cdots\cdots \twoheadrightarrow A/\mathfrak{m}^2 \twoheadrightarrow A/\mathfrak{m} = k$. $\phi(k)$: always bij. $\square$

Def. $\cdot$ $R \in \hat{\mathcal{C}}$. $x \in \hat{F}(R)$. $\Rightarrow x^* : \mathrm{Spf}\, R \longrightarrow F$

$$x^*(A) : \mathrm{Spf}\, R(A) = \mathrm{Hom}_{\hat{\mathcal{C}}}(R, A) \ni \psi \longmapsto \hat{F}(\psi)(x) \in F(A). \quad (\forall A \in \mathcal{C})$$

$\cdot$ $f : R \to R'$. $\hat{F}(f) : x \longmapsto x'$. $\Rightarrow x'^* = x^* \circ f^*$ $[\hat{F}(\psi)(x') = \hat{F}(\psi \circ f)(x) = \hat{F}(f^*\psi)(x)]$

Th'm F: formally smooth. $\dim_k TF(k) = n \Rightarrow \exists \phi : \mathrm{Spf}\, \mathcal{O}_K[[X_1, \dots, X_n]] \xrightarrow{\sim} F$.

∵) $X := \bigoplus_{i=1}^{n} k \cdot X_i$

$$\hat{F}(\mathcal{O}_K[[X_1, \dots, X_n]]) \longrightarrow F(k[X])$$
$$\exists \tilde{x} \longmapsto x \in TF(X)$$

$\Rightarrow \phi := \tilde{x}^*$.

$$\begin{tikzcd} \mathrm{Hom}_k(X, -) = T\mathrm{Spf}\, k[X] \arrow[rd, "Tx^*"] \arrow[d, "\wr", "Tf^*"'] & \\ T\, \mathrm{Spf}\, \mathcal{O}_K[[X_1, \dots, X_n]] \arrow[r, "T\tilde{x}^*"'] & TF \end{tikzcd} \quad \circlearrowright$$

enough to choose $x \in TF(X)$ s.t. $Tx^* : \mathrm{Hom}_k(X, -) \xrightarrow{\sim} TF$.

$$TF(X) \cong \bigoplus_{i=1}^{n} TF(k \cdot X_i) \xrightarrow{\sim} TF(k)^n.$$
$$x = (x_1, \dots, x_n) \longmapsto (\varphi_1(x_1), \dots, \varphi_n(x_n))$$

where. $\left(k \cdot X_i \ni X_i \xrightarrow{\sim} 1 \in k, \quad TF : TF(k \cdot X_i) \xrightarrow[\varphi_i]{\sim} TF(k) \right)$.

$$Tx^*(k) : \mathrm{Hom}_k(X, -) \xrightarrow{TF} \mathrm{Hom}_k(TF(X), TF(k)) \xrightarrow{\text{eval. at } x} TF(k)$$

Basis: $X_i^* := \begin{pmatrix} X_i \mapsto 1 \\ X_j \mapsto 0 \end{pmatrix} \mapsto TF(X_i^*) = (0, \dots, \varphi_i, \dots, 0) \longmapsto \varphi_i(x_i) \in$

$\cong$ if $\varphi_1(x_1), \dots, \varphi_n(x_n)$: basis of $TF(k)$. $\square$