※ Weil gp.

Thm 1 (Lubin-Tate theory).
1) $\exists K^{LT} \subset K^{ab}$ [s.t. $K^{LT} = K^{ur} K_{\varpi}^{ram}$ $(\forall \varpi)$]
$\exists !$ hom. $K^{\times} \xrightarrow{\mathrm{Art}_K} \mathrm{Gal}(K^{LT}/K)$ s.t.

$\forall \varpi \in K$: unif. $\mathrm{Art}_K(\varpi) = \begin{cases} \mathrm{Frob}_K & \text{on } K^{ur} \\ \mathrm{id} & \text{on } K_{\varpi}^{ram} \end{cases}$.

2). $K^{\times} \xrightarrow{\sim} W_K^{LT} := W(K^{LT}/K) \subset \mathrm{Gal}(K^{LT}/K)$

Lem (uniqueness) If you have a hom $K^{\times} \xrightarrow{\mathrm{Art}'_K} \mathrm{Gal}(K^{LT}/K)$ s.t.

i). $\forall \varpi$: unif. $\mathrm{Art}'_K(\varpi)|_{K^{ur}} = \mathrm{Frob}_K$.
ii) $\forall K'/K$ fin. $K' \subset K^{LT}$. $\mathrm{Art}'_K(N_{K'/K}(x))|_{K'} = \mathrm{id}$ $(\forall x \in K'^{\times})$
$\Rightarrow \mathrm{Art}'_K = \mathrm{Art}_K$.

∵). need $\mathrm{Art}_K(\varpi)|_{K_{\varpi}^m} = \mathrm{id}$ $(\forall m \geq 1)$. but $\varpi = N_{K_{\varpi}^m/K}(-\alpha)$. //

Prop. 1 (Base Change). Art_K satisfies ii).
Prop. 2 (Local Kronecker-Weber). $K^{LT} = K^{ab}$. [$\Leftrightarrow$ Hasse-Arf]

$\Rightarrow$ **Thm 2** (LCFT). 1) $\exists !$ hom $K^{\times} \xrightarrow{\mathrm{Art}_K} \mathrm{Gal}(K^{ab}/K)$ s.t.

i) $\forall \varpi$: unif. $\mathrm{Art}_K(\varpi)|_{K^{ur}} = \mathrm{Frob}_K$.
ii) $\forall K'/K$ fin. abel. $\mathrm{Art}_K(N_{K'/K}(x))|_{K'} = \mathrm{id}$ $(\forall x \in K'^{\times})$
2) $K^{\times} \xrightarrow{\sim} W_K^{ab} := W(K^{ab}/K)$.

Thm 3 (Galois Cohomology). $\exists \mathrm{Art}'_K : K^{\times} \to \mathrm{Gal}(K^{ab}/K)$ s.t.
i). ii) for $\forall K'/K$ fin. abel, iii) $K^{\times} \twoheadrightarrow W_K^{ab}$.

(LT) + (GC) $\Rightarrow$ [LCFT] ∵). $K^{\times} \xrightarrow{\mathrm{Art}'_K} W_K^{ab} \twoheadrightarrow W_K^{LT}$, composite $\sim$ Art_K. (Apply Lemma to the composite.)

(LT) + [Base Change] + [Local K-W $K^{LT} = K^{ab}$] $\Rightarrow$ [LCFT]
(GC) + [Uniqueness] + [Existence thm $K^{\times} \hookrightarrow W_K^{ab}$] $\Rightarrow$ [LCFT]

K. $\mathcal{O}$. $\mathfrak{p}$. $k := \mathcal{O}/\mathfrak{p} \cong \mathbb{F}_q$

Def. A: $\mathcal{O}$-alg. Formal $\mathcal{O}$-module $/A$

… $\Sigma = (F, [\cdot])$. $\begin{cases} F(X, Y) \in A[[X, Y]] : \text{formal gp.} \\ [\cdot] : \mathcal{O} \to \mathrm{End}(F) : \text{ring hom.} \end{cases}$

s.t. $[a](X) \equiv aX \pmod{\deg 2}$.

[precisely. $i(a)$. $i: \mathcal{O} \to A$]

$(F, [\cdot])$ $(F', [\cdot]')$
$\Sigma, \Sigma' /A$: formal $\mathcal{O}$-mod.

$\mathrm{Hom}(\Sigma, \Sigma') \ni f$: homomorphism $f(X) \in (X) \subset A[[X]]$.

$\Leftrightarrow \begin{cases} f : \text{hom of formal gps} \quad (f \circ F = F' \circ f) \\ f \circ [a] = [a]' \circ f. \quad (\forall a \in \mathcal{O}). \end{cases}$

$\mathcal{O}$-mod. by $+_{F'}$, $[\cdot]'$

$\mathrm{End}(\Sigma)$: (non-commutative) $\mathcal{O}$-alg. by $+_F$, $\circ$.

Ex. $f \equiv \begin{cases} \varpi X & \bmod X^2 \\ X^{q^n} & \bmod \mathfrak{p} \end{cases} \rightsquigarrow \Sigma_f := (F_f, [\cdot]_f) /\mathcal{O}_L$

$\in \mathcal{O}_L[[X]]$. (L/K comp. unram.) … formal $\mathcal{O}$-mod.

$\theta \in \mathcal{O}_L$ s.t. $\theta^{\varphi^n}/\theta = \varpi'/\varpi \rightsquigarrow [\theta] := [\theta]_{f,f'} : \Sigma_f \to \Sigma_{f'}$

… hom of formal $\mathcal{O}$-mod's.

We will see: Lubin-Tate gps$/\mathcal{O}_L$ = formal $\mathcal{O}$-mod. of height $1/\hat{\mathcal{O}}_L$.
unique up to isom / $\hat{\mathcal{O}} = \mathcal{O}_{\hat{K}^{ur}}$.

- formal $\mathcal{O}$-mod of height $n/\overline{\mathbb{F}}_q$: unique up to isom.
- ——— " ——— $/\hat{\mathcal{O}}$: not unique when $n > 1$.

We will study:

- formal $\mathcal{O}$-mod's $/\overline{\mathbb{F}}_q$. $/\mathcal{O}_L$. $/A$: complete noetherian local $\mathcal{O}$-alg. w/ res. field ($\mathbb{F}_q$ or) $\overline{\mathbb{F}}_q$

∘ A, B : $\mathcal{O}$-alg. $A \xrightarrow{i} B$: $\mathcal{O}$-alg. hom.

$\Sigma/A \rightsquigarrow \Sigma \otimes_A B / B$: functor. $\mathrm{Hom}_A(\Sigma, \Sigma') \longrightarrow \mathrm{Hom}(\Sigma \otimes B, \Sigma' \otimes B)$

$f \longmapsto i(f)$

$i: A[[X]] \longrightarrow B[[X]]$.

- Classification (up to isom …)
- Structure of $\mathrm{Hom}(\Sigma, \Sigma')$, $\mathrm{End}(\Sigma)$ $\quad \Sigma[\mathfrak{p}^m] \simeq (\mathcal{O}/\mathfrak{p}^m)^h$.

formal gps $/\mathbb{F}_p$ = always formal $\mathbb{Z}_p$-module.

Prop. A: ring. F, G: formal gp$/A$. $f: F \to G$.

i) If A: $\mathbb{Z}$-torsion free, then $f'(0)=0 \Rightarrow f=0$. In particular $End(F) \hookrightarrow A$, $f \mapsto f'(0)$.

ii) If A: $\mathbb{F}_p$-alg, then $f \neq 0 \Rightarrow \exists!\, h \geq 0$ s.t. $\begin{cases} f(x) = g(X^{p^h}) \\ g'(0) \neq 0 \end{cases}$.

$h := ht(f)$: <u>height</u> of f. $ht(0) := \infty$

[f: isom $\Rightarrow ht(f)=0$; $\Leftarrow$ if A: field]

pf. $f \circ F = G \circ f \underset{\partial/\partial Y}{\Longrightarrow} (f' \circ F)\cdot F_Y = (G_Y \circ f)\cdot f'(Y)$

<u>Assume $f'(0)=0$</u>. set $Y=0$ $\begin{cases} RHS = 0 \\ LHS = f'(x)\cdot F_Y(x,0) \end{cases}$ [$\because$ $F(x,0)=X$]

$F(X,Y) \equiv X+Y$ mod deg 2.

$F_Y(x,0) \equiv 1 \mod X^2 \Rightarrow \in A[[X]]^\times$. i.e. $f'(x)=0$ in $A[[X]]$.

- A: $\mathbb{Z}$-tors free $\Rightarrow f=0$.
- A: $\mathbb{F}_p$-alg. ($\Rightarrow p \nmid n \Rightarrow n \in \mathbb{F}_p^\times \subset A^\times$) $\Rightarrow f(x) = g(x^p)$. $\Rightarrow$ if $f \neq 0$, repeat until $g'(0) \neq 0$ ∎

[when f: isom, $ht(f\circ g) = ht(g)$.]

Lem. A: $\mathbb{F}_p$-alg.
i) $ht(f \circ g) \geq ht(f) + ht(g)$ [$=$ if A: domain]
ii) $ht(f \underset{G}{+} g) \geq \min(ht(f), ht(g))$
iii) $ht([p]) \geq 1$ ($\because$ $[p](X) \equiv pX = 0 \mod X^2$)

Cor. A: $\mathbb{F}_p$-alg. $F, G/A$.
$H := Hom_A(F,G) = H^0 \supset H^1 \supset H^2 \supset \cdots$
$H^m := \{ f \in H \mid ht(f) \geq m \}$. ($\forall m \geq 0$)

$\Rightarrow$ H^m: subgp by $+_G$ (ii), $[p^m]_G \cdot H \subset H^m$ (i),(iii).
$H \xrightarrow{\sim} \varprojlim H/H^m$: $\mathbb{Z}_p$-mod. [$\because$ H/H^m: $\mathbb{Z}/p^m$-mod.]

In particular, $\mathbb{Z}_p \xrightarrow{\text{ring hom}} End(F)$.

Prop. A: $k = \mathcal{O}/\mathfrak{p}$-alg ($k \cong \mathbb{F}_q$). Σ, Σ': formal $\mathcal{O}$-mod.$/A$ [A: $\mathcal{O}$-alg. by $\mathcal{O} \twoheadrightarrow k$].

$f: \Sigma \to \Sigma'$ ($f \neq 0$) $\Rightarrow \exists!\, h \geq 0$. $f(x) = g(X^{q^h})$ $g'(0) \neq 0$.

$h := ht_{\mathcal{O}}(f)$: <u>($\mathcal{O}$-) height</u> of f. [if $q = p^b$. (height as fml gp) $=$ $b\cdot$($\mathcal{O}$-height)]

$\because$) $f = g(X^{p^{h'}})$. $g'(0) \neq 0$. $f \equiv g'(0)\cdot X^{p^{h'}}$ (mod deg $p^{h'}+1$).

$\left.\begin{array}{l} f \circ [a] \equiv g'(0)\cdot(aX)^{p^{h'}} \\ [a]' \circ f \equiv g'(0)\cdot a \cdot X^{p^{h'}} \end{array}\right\}$ mod deg $p^{h'}+1$. $\Rightarrow g'(0)\cdot(\underbrace{a^{p^{h'}} - a}) = 0$ in A ($\forall a \in \mathcal{O}$). ($k \subset A$.)

If $p^{h'} \neq q^h$ for some h. $\exists a \in \mathcal{O}$ s.t. $a^{p^{h'}} - a \in k^\times \Rightarrow g'(0) = 0$. ✗. ∎

Def. <u>($\mathcal{O}$-) height</u> of Σ/A: k-alg. $:= ht([\varpi]_F)$. $\varpi \in \mathcal{O}$: unif.
indep. of ϖ by Lemma i).