Lecture 3. 12/12/07

Ex. $K = \mathbb{Q}_p$. $p \in \mathbb{Z}_p$. $f(x) = (1+X)^p - 1 \Rightarrow F_f(X,Y) = \hat{\mathbb{G}}_m(X,Y)$
$= (1+X)(1+Y) - 1$.

$m \geq 0$. $f_m(x) = \underbrace{f \circ \cdots \circ f}_{m} = (1+X)^{p^m} - 1 \Rightarrow \mu_{f,m} = \{ \zeta - 1 \mid \zeta \in \mu_{p^m} \}$.

$\Rightarrow L(\mu_{f,m}) = L(\mu_{p^m})$. $L/\mathbb{Q}_p$: complete unram.

$$\mathrm{Gal}(L(\mu_{p^m})/L) \xrightarrow{\sim} \mathrm{Aut}_{\mathbb{Z}_p}(\mu_{f,m}) \simeq (\mathbb{Z}/p^m)^\times$$

$$(\zeta \mapsto \zeta^u) \underset{\uparrow}{=} (\alpha \mapsto [u](\alpha) \quad (\forall \alpha \in \mu_{f,m})) \mapsto u \bmod p^m$$

$[\, u \in \mathbb{Z}$. $[u]_f(X) = (1+X)^u - 1 \Rightarrow [u](\zeta - 1) = \zeta^u - 1$. $]$

Say $n = n' \cdot p^m$
v : a place of $\mathbb{Q}(\mu_{n'}) =: F$ above p, $L = F_v$.

$$\mathrm{Gal}(L(\mu_{p^m})/L) \xrightarrow{\sim} (\mathbb{Z}/p^m)^\times$$

$$\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/n)^\times$$

$\to$ inertia gp of v.

$(\zeta \mapsto \zeta^u) \mapsto u \bmod n$

global Artin map. $\mathrm{Fr}_\ell \mapsto \ell \bmod n \quad \forall \ell \nmid n$.

$\cdots$ compatible w/ the local Artin map.

Rem. $\mathbb{Q}(\mu_n)$ [and possibly $F/\mathbb{Q}$ imag. quad. + torsion of CM ell. curves] is the only case where you see the local-global compatibility explicitly.
General CFT : reduce to this case. GL_n : reduce to CFT (GL_1).

L/K : complete unram. $\varpi \in L$. $f \in \mathcal{O}_L[X]$, $\begin{cases} \equiv \varpi X & \bmod X^2 \\ \equiv X^q & \bmod \mathfrak{p} \end{cases}$.

$m \geq 1$.

③ $\rho_{f,m}$:

$$\begin{array}{ccccc} \mathrm{Gal}(L(\mu_{f,m})/L) & \xrightarrow{\sim} & \mathrm{Aut}_{\mathcal{O}}(\mu_{f,m}) & = & (\mathcal{O}/\mathfrak{p}^m)^\times \\ \uparrow \text{res.} & & \text{res.} \uparrow & & \uparrow \\ \mathrm{Gal}(L(\mu_{f,m'})/L) & \xrightarrow{\sim} & \mathrm{Aut}_{\mathcal{O}}(\mu_{f,m'}) & = & (\mathcal{O}/\mathfrak{p}^{m'})^\times \end{array}$$

$m \leq m'$.
$\mu_{f,m} \subset \mu_{f,m'}$.

② $[\theta]_{f,f'} \in \mathrm{Hom}_{\mathcal{O}_L}(F_f, F_{f'})$. $\theta \in \mathcal{O}_L^\times \Rightarrow [\theta] : \mu_{f,m} \xrightarrow{\sim} \mu_{f',m}$
$\alpha \mapsto [\theta](\alpha)$.

$$\begin{array}{ccc} \mathrm{Gal}(L(\mu_{f,m})/L) & \xrightarrow{\sim} & \mathrm{Aut}_{\mathcal{O}}(\mu_{f,m}) \\ \| & & \downarrow\wr \\ \mathrm{Gal}(L(\mu_{f',m})/L) & \xrightarrow{\sim} & \mathrm{Aut}_{\mathcal{O}}(\mu_{f',m}) \end{array}$$

$[u\theta](\alpha)$.
$\quad\quad ''$
$([\theta](\alpha) \mapsto [u]([\theta](\alpha)))$
$= (\alpha \mapsto [u](\alpha))$

① $\mathrm{Gal}(L(\mu_{f,m})/L) \longrightarrow \mathrm{Aut}_{\mathcal{O}}(\mu_{f,m})$ [Galois gp respects the $\mathcal{O}$-mod. str. def'd by power series$/\mathcal{O}_L$.]

$\boxed{\text{Galois action}} \longleftrightarrow \boxed{\text{"Geometric" (def'd/base field) action}}$ .... " $\mu_{f,m}$: $\mathcal{O}$-module scheme $/\mathcal{O}_L$ "
(group).

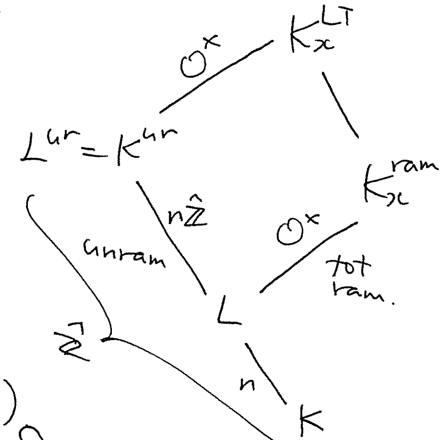
· $\underline{L = \widehat{K^{ur}}}$ $\quad \mu_{f,m} \xrightarrow[\sim]{\exists[\theta]} \mu_{f',m}$ for $\forall f, f'$.

$\Rightarrow \widehat{K}^{m} := \widehat{K}^{ur}(\mu_{f,m})$.

· $\underline{L/K \text{ fin}}$, $[L:K]=n$. $\quad \mu_{f,m} \xrightarrow[\sim]{[\theta]} \mu_{f',m}$ if $N_{L/K}(\varpi) = N_{L/K}(\varpi') =: x \in K^\times$, $v(x)=n$.

$\Rightarrow K_x^m := L(\mu_{f,m})$. $\quad K_x^{ram} := \bigcup_{m\geq 1} K_x^m$.

$$\begin{tikzcd}
\rho_{f,m} : \mathrm{Gal}(K_x^m/L) \arrow[r, "\sim"] & (\mathcal{O}/\mathfrak{p}^m)^\times \\
\rho_x : \mathrm{Gal}(K_x^{ram}/L) \arrow[r, "\sim"] \arrow[u, two heads] & \mathcal{O}^\times \arrow[u, two heads]
\end{tikzcd}$$



$K_x^{LT} := K_x^{ram}K^{ur}$. $\quad K_x^{ram} \cap K^{ur} = L$

$$\begin{tikzcd}
\mathrm{Gal}(K_x^{LT}/K) \simeq \mathrm{Gal}(K_x^{ram}/L) \times \mathrm{Gal}(K^{ur}/L) \arrow[r, phantom, "\subset"] & \mathrm{Gal}(K^{ur}/K) \ni \mathrm{Frob}_K \arrow[d] \\
\mathcal{O}^\times \times n\hat{\mathbb{Z}} \arrow[r, hook] & \hat{\mathbb{Z}} \ni 1
\end{tikzcd}$$

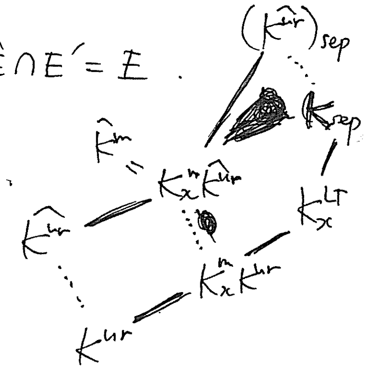
$\underline{\text{Thm 1}}$. $K_x^m K^{ur}$, is independent of x.
$\left(\Rightarrow K^{LT} := K_x^{LT} = \bigcup_{m\geq 1} K_x^m K^{ur} \right)$

$\underline{\text{pf}}$ $\quad K_x^m = L(\mu_{f,m})$. $\quad K_{x'}^m = L'(\mu_{f',m})$

$K_x^m \widehat{K^{ur}} = \widehat{K^{ur}}(\mu_{f,m}) = \widehat{K^{ur}}(\mu_{f',m}) = K_{x'}^m \widehat{K^{ur}}$.

$\underline{\text{Lemma}}$ $\left.\begin{array}{l} E/E\cap K^{ur} : \text{fin} \\ E/K : \text{sep.} \end{array}\right\}$ E'/E fin sep. $\Rightarrow E'\hat{E} = \widehat{E'}$. $\hat{E}\cap E' = E$.
(in particular: $\underbrace{\widehat{E'} = \widehat{E''}}_{\hat{E}\cap E_{sep} = E} \Rightarrow E' = E''$).

$\|$.



$$\begin{array}{ccc} K_x^m \widehat{K^{ur}} & = & \widehat{K_x^m K^{ur}} \\ \| & & \| \\ K_{x'}^m \widehat{K^{ur}} & = & \widehat{K_{x'}^m K^{ur}} \end{array} \quad \Rightarrow \quad K_x^m K^{ur} = K_{x'}^m K^{ur}.$$

∎

$x \in K^\times$. $v(x) = n \geq 1$.

Lem. $\exists \varpi \in L$. L/K : fin unram. $[L:K]=n$. $N_{L/K}(\varpi) = x$.
$(N_{L/K} : \mathcal{O}_L^\times \twoheadrightarrow \mathcal{O}_K^\times)$.

$\Rightarrow K_x^m$. K_x^{ram}. K_x^{LT}. ρ_x $\left(\alpha \mapsto [u](\alpha) \text{ on } \mu_{f,m}\right)$ $\forall m$.

$Art_K^x : \mathcal{O}^\times \times \langle x \rangle \ni u \cdot x^b \longmapsto (\rho_x^{-1}(u).\ Frob_L^b) \in Gal(K^{LT}/L)$

$\mathcal{O}^\times \times \langle x \rangle = K^\times \supset v^{-1}(n\mathbb{Z})$

$Gal(K^{LT}/L) \cong Gal(K_x^{ram}/L) \times Gal(K^{ur}/L)$.

Th'm 2. Art_K^x : "indep. of x." i.e. :

$v(x) = 1$: $Art_K : K^\times \longrightarrow Gal(K^{LT}/K)$: indep. of x.

$v(x) = n$: $Art_K^x = Art_K|_{v^{-1}(n\mathbb{Z})}$.

pf. $v(x) = v(x') = n$.

Claim $Art_K^{x'}(x') = Art_K^x(x')$.

$\begin{cases} K_x^m = L(\mu_{f,m}) \\ K_{x'}^m = L(\mu_{f',m}) \end{cases}$

$\begin{cases} \text{id on } K_{x'}^{ram} \\ Frob_K^n \text{ on } K^{ur} \end{cases}$ $\quad$ $\begin{cases} \alpha \mapsto [x'/x]_f(\alpha) \ (\alpha \in \mu_{f,m}). \text{ on } K_x^{ram} \\ Frob_K^n \text{ on } K^{ur}. \end{cases}$

$\mu_{f',m} \xleftarrow[{[\theta]/\mathcal{O}_{\widehat{K^{ur}}}}]{\sim} \mu_{f,m} \ni \alpha$

$\boxed{Frob_L = Frob_K^n}$.

on $K_{x'}^{ram} = L(\mu_{f',m})$:

$Art_K^x(x') : [\theta](\alpha) \longmapsto [\theta]^{\varphi^{-n}}\left([x'/x]_f(\alpha)\right) = \left([\theta] \circ [x'/x]_f\right)^{\varphi^{-n}}(\alpha)$ — computation in $\hat{K}^m = K_x^m \hat{K}^{ur}$

i.e. id on $K_{x'}^{ram}$. $= [\theta](\alpha)$ $\qquad [\theta] \circ [x'/x]_f = [\theta]^{\varphi^n}$ Last time Prop. 2.

$\Rightarrow$ $Art_K^x(x'') = Art_K^{x''}(x'') = Art_K^{x'}(x'')$ $(\forall x''.\ v(x'') = n)$

$Art_K^x = Art_K^{x'}$ $\qquad$ generates $v^{-1}(n\mathbb{Z}) \subset K^\times$.

$n = 1$. get first line.

$v(x) = 1 \Rightarrow v(x^n) = n$. $Art_K^{x^n} = Art_K^x|_{v^{-1}(n\mathbb{Z})}$ by def.

Cor. $Art_K : K^\times \xrightarrow{\sim} W(K^{LT}/K) \subset Gal(K^{LT}/K)$. $Art_K(x)$ $v(x) > 0$

$W(K^{LT}/K) = \{\sigma \mid \sigma|_{K^{ur}} \in Frob_K^{\mathbb{Z}}\}$.

characterized by $\begin{cases} Frob_K^{v(x)} \text{ on } K^{ur} \\ \text{id on } K_x^{ram} \end{cases}$