Lecture 12. 2/11/07.

- Group action on $\mathcal{F}_m$.

$\mathcal{D}_0^\times := \mathrm{Aut}_K(\Sigma_0) \ni d$. $\quad [(\Sigma . i . \alpha)] \longmapsto [(\Sigma . i\circ d . \alpha)]$

$GL_n(\mathcal{O}/\mathfrak{p}^m) = \mathrm{Aut}_{\mathcal{O}}((\mathfrak{p}^{-m}/\mathcal{O})^n) \ni g$. $\quad [(\Sigma . i . \alpha)] \longmapsto [(\Sigma . i . \alpha\circ g)]$

gives $\mathcal{D}_0^\times \times GL_n(\mathcal{O}/\mathfrak{p}^m) \overset{\text{right}}{\curvearrowright} \mathcal{F}_m$.

Lem. $\forall a \in \mathcal{O}^\times$. $(a. \ a\cdot 1_n \bmod \mathfrak{p}^m) \in \mathcal{D}_0^\times \times GL_n(\mathcal{O}/\mathfrak{p}^m)$ acts trivially on $\mathcal{F}_m$.

$\because$) $[a] : \Sigma \xrightarrow{\sim} \Sigma$. gives $(\Sigma . i . \alpha) \xrightarrow{\sim} (\Sigma, \underset{i\circ a}{\underset{\|}{[a]\circ i}}, \underset{\alpha\circ a}{\underset{\|}{[a]\circ\alpha}})$ $\lrcorner$

Can consider $\{\mathcal{F}_m\}_{m\ge 0}$ all at once. or the inverse limit

[tower $\cdots \to \mathcal{F}_m \to \mathcal{F}_{m-1} \to \cdots \to \mathcal{F}_0$] $\quad \mathcal{F} := \varprojlim_m \mathcal{F}_m$.

$GL_n(\mathcal{O}/\mathfrak{p}^m) \twoheadrightarrow GL_n(\mathcal{O}/\mathfrak{p}^{m-1})$ $\quad$ def'd by $\mathcal{F}(A) = \varprojlim_m \mathcal{F}_m(A)$ $(\forall A \in \hat{\mathcal{C}})$.

- as $\varprojlim_m \mathrm{Hom}_{\hat{\mathcal{C}}}(R_m . A) = \mathrm{Hom}(\varinjlim R_m . A)$.
  $\mathcal{F}$ is "ind-rep'ed" by $R := \varinjlim R_m$ (not noetherian any more!)
- an elt of $\mathcal{F}(A) = \varprojlim_m \mathcal{F}_m(A)$ is an isom class of $(\Sigma . i . (\alpha_m)_{m\ge 0})/A$.
  where $\alpha_m : (\mathfrak{p}^{-m}/\mathcal{O})^n \longrightarrow \mathfrak{m}_\Sigma$, $\alpha_{m'}|_{(\mathfrak{p}^{-m}/\mathcal{O})^n} = \alpha_m$ for $m' \ge m$.

  $\Longleftrightarrow$ giving $\alpha : (K/\mathcal{O})^n \xrightarrow[\mathcal{O}\text{-hom.}]{} \mathfrak{m}_\Sigma$ s.t. $[\mathfrak{p}] = \left(\prod_{x\in(\mathfrak{p}^{-1}/\mathcal{O})^n}(X - \alpha(x))\right)$ in $A[\![X]\!]$.

  [because $K/\mathcal{O} = \bigcup_{m\ge 0} \mathfrak{p}^{-m}/\mathcal{O}$]. $\quad \alpha = (\alpha_m)_{m\ge 0}$: "level $\mathfrak{p}^\infty$-str".

$\Rightarrow \mathcal{D}_0^\times \times GL_n(\mathcal{O}) \overset{\text{right}}{\curvearrowright} \mathcal{F}$.

by Lemma. this action factors through $(\mathcal{D}_0^\times \times GL_n(\mathcal{O}))/\mathcal{O}^\times$ diagonally embedded.

- Now we will "enlarge" the functors $\mathcal{F}$. $\mathcal{F}_m$
  to make it carry an action of bigger group $(D_0^\times \times GL_n(K))/K^\times$.
  [$D_0 = \mathcal{D}_0 \otimes_{\mathcal{O}} K$. $\mathcal{D}_0 := \mathrm{End}_K(\Sigma_0)$].

Defining this action amounts to
defining Hecke correspondence in our situation.

∘ Quasi-isogenies between formal $\mathcal{O}$-mod $/\kappa$.

Lem. Σ, Σ': formal $\mathcal{O}$-mod$/\kappa$. $f : \Sigma \to \Sigma'$. hom. $f \neq 0$ $\left[\Rightarrow \begin{array}{c} \mathrm{ht}\,\Sigma \\ \| \\ \mathrm{ht}\,\Sigma' \\ \| \\ n \end{array} \right]$.
$\Rightarrow \exists m.\ \exists f' : \Sigma' \to \Sigma$ s.t. $f' \circ f = [\varpi^m]$.

∵) $f = u \circ X^{q^h}$ ($h = \mathrm{ht}\, f \geq 0$). $[\varpi^m] = u' \circ X^{q^{mn}}$ ($u, u' \in \kappa[\![X]\!]^\times$).
choose m s.t. $mn \geq h$. $f' := u' \circ X^{q^{mn-h}} \circ u^{-1}$ will do. ┘

Cor. Σ_0 : formal $\mathcal{O}$-mod$/\kappa$. $\mathcal{D}_0 := \mathrm{End}_\kappa(\Sigma_0)$: $\mathcal{O}$-alg. (non-commutative in general)
$\Rightarrow$ $D_0 :=$ $\mathcal{D}_0 \otimes_{\mathcal{O}} K$ (… K-alg) is a division alg.

∵) $^\forall$elt of D_0 is of the form $f \otimes \varpi^m$ ($f \in \mathcal{D}_0$. $m \in \mathbb{Z}$). ϖ: unif of K.
multiplication is given by $(f \otimes \varpi^m) \cdot (g \otimes \varpi^{m'}) = (f \circ g) \otimes \varpi^{m+m'}$. use Lemma. ┘

Def. Σ, Σ' : formal $\mathcal{O}$-mod$/\kappa$. $f \in \underbrace{\mathrm{Hom}_\kappa(\Sigma, \Sigma') \otimes_{\mathcal{O}} K}_{K\text{-vect.sp}}$. $f \neq 0$. is called a quasi-isogeny from Σ to Σ'.
For $f \in \mathrm{Hom}(\Sigma, \Sigma')$. define $\mathrm{ht}(f \otimes \varpi^m) = \mathrm{ht}(f) + m \cdot \underbrace{\mathrm{ht}([\varpi])}_{= n}$. ($\forall m \in \mathbb{Z}$).
$\Rightarrow$ can define $\begin{cases} (f \otimes \varpi^m) \circ (g \otimes \varpi^{m'}) = (f \circ g) \otimes \varpi^{m+m'}. \\ \mathrm{ht}(f \circ g) = \mathrm{ht}(f) + \mathrm{ht}(g) \ (\Leftarrow {}^\forall f, g : q\text{-isog}) \end{cases}$ $\boxed{\begin{array}{l} \mathrm{ht} = 0 \Leftrightarrow \text{isom.} \\ \mathrm{ht} \geq 0 \Leftrightarrow \text{hom.} \end{array}}$

Ex. Q-isog from Σ_0 to Σ_0 = non zero elt of D_0. … multiplicative gp $D_0^\times$ by Cor. $D_0^\times = D_0 \setminus \{0\}$.

We modify the def'n of $\mathcal{F}_m$.

Def. (NEW $\mathcal{F}_m$). $\left[\begin{array}{c} m \geq 1. \\ \Sigma_0/\kappa. \end{array} \right]$ $\mathcal{F}_m : \hat{\mathcal{C}} \longrightarrow (\mathrm{Sets})$.
$(A, \mathfrak{m}) \longmapsto \{ (\Sigma, i, \alpha) \mid \Sigma, \alpha$ as before. $i : \Sigma_0 \to \Sigma \bmod \mathfrak{m}$ quasi-isog. [instead of isom] $\} / \cong$
$\forall j \in \mathbb{Z}$. $\mathcal{F}_m^{(j)} : (A, \mathfrak{m}) \longmapsto \{ (\Sigma, i, \alpha) \mid \mathrm{ht}(i) = j \} / \cong$.
… subfunctor of $\mathcal{F}_m$.
$\Rightarrow$ $\mathcal{F}_m = \coprod_{j \in \mathbb{Z}} \mathcal{F}_m^{(j)}$. $\mathcal{F}_m^{(0)} = \left(\text{our old } \mathcal{F}_m \right)$.

Rem. We will see the representability of $\mathcal{F}_m^{(j)}$ next week.

Now we have: $d \in D_0^\times \overset{\text{right}}{\curvearrowright} \mathcal{F}_m$ by $[(\Sigma, i, \alpha)] \xmapsto{[d]} [(\Sigma, i \circ d, \alpha)]$.
[if $\mathrm{ht}(d) = h$. then $\mathcal{F}_m^{(j)} \xrightarrow[{[d]}]{} \mathcal{F}_m^{(j+h)}$]