# UNITARY SHIMURA VARIETIES I

## 1. Polarized Abelian Varieties over $\mathbb{C}$

An *abelian variety* $A$ over $\mathbb{C}$ is a smooth projective group variety over $\mathbb{C}$. The complex points $A(\mathbb{C})$ have the structure of a *complex torus*, that is, a complex manifold isomorphic to $W/\Lambda$ where $W$ is a complex vector space of dimension $g$, say, and $\Lambda$ is a $\mathbb{Z}$-lattice in $W$. If $X = W/\Lambda$ is a complex torus then we have a canonical isomorphisms

$$\operatorname{Lie} X/H_1(X,\mathbb{Z}) \ \tilde{\rightarrow}\ X \tag{1.1}$$

$$H^i(X,\mathbb{Z}) \cong \operatorname{Hom}(\wedge^i H_1(X,\mathbb{Z}),\mathbb{Z}) \tag{1.2}$$

$$H^i(X,R) \cong H^i(X,\mathbb{Z})\otimes_{\mathbb{Z}} R \tag{1.3}$$

for any ring $R$. The first isomorphism arises as follows: the exponential map defines a surjection

$$\operatorname{Lie} X \to X$$

which allows us to regard $\operatorname{Lie} X$ as the universal cover of $X$. The kernel of this map is a lattice in $\operatorname{Lie} X$ which we can identify with the fundamental group of $X$ or indeed $H_1(X,\mathbb{Z})$ since it is abelian. In particular we have an isomorphism of $\mathbb{R}$-vectors spaces $\operatorname{Lie} X \cong H_1(X,\mathbb{R})$.

Not every complex torus arises as the $\mathbb{C}$ points of an abelian variety, only those which are *polarizable*. By definition, a complex torus $X = W/\Lambda$ is said to be polarizable if there exists a Hermitian form (with $\mathbb{C}$-linearity in the second variable)

$$H : W \times W \to \mathbb{C}$$

which is

(i) non-degenerate
(ii) positive definite
(iii) such that $\operatorname{Im} H(\Lambda,\Lambda) \subset \mathbb{Z}$.

A *polarization* on a complex torus $X$ is a choice of such a form $H$, if one exists.

There is a bijection between Hermitian forms $H : W \times W \to \mathbb{C}$ and alternating $\mathbb{R}$-linear forms $E : W \times W \to \mathbb{R}$ such that $E(iu, iv) = E(u, v)$ for all $u, v \in W$. The relation is

$$H(u, v) = E(u, iv) + iE(u, v).$$

Moreover, $H$ satisfies $(i), (ii)$ and $(iii)$ if and only if $E$ is

(i) non-degenerate
(ii) such that the symmetric form $(u, v) \mapsto E(u, iv)$ is positive definite
(iii) integer valued on $\Lambda \times \Lambda$.

How do polarizations arise? The exponential exact sequence gives rise to a map

$$c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X,\mathbb{Z}) \cong \operatorname{Hom}(\wedge^2\Lambda,\mathbb{Z}).$$

The image of this map consists of alternating forms $E : \Lambda \times \Lambda \to \mathbb{Z}$ such that the $\mathbb{R}$-linear extension of $E$ satisfies $E(iu, iv) = E(u, v)$ for all $u, v \in \operatorname{Lie}(X)$. When does a holomorphic line bundle $\mathcal{L}$ give rise to a form $E = c_1(\mathcal{L}) \otimes \mathbb{R}$ which satisfies $(i)$ and $(ii)$ above? To answer this, we need to introduce the dual torus of $X$.

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If $X = W/\Lambda$ is a complex torus, we define the dual complex torus as follows. First of all, let $\widehat{\overline{W}}$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-antilinear forms $W \to \mathbb{C}$. Let $\widehat{\Lambda} = \{l \in \widehat{\overline{W}} : \operatorname{Im} l(\Lambda) \subset \mathbb{Z}\}$. Then we let $\hat{X} = \widehat{\overline{W}}/\hat{\Lambda}$.

Let $\mathcal{L}$ be a holomorphic line bundle $X$ and let $H$ be the Hermitian form on $W$ corresponding to $c_1(\mathcal{L})$. Then there is a holomorphic morphism $\varphi_{\mathcal{L}} : X \to \hat{X}; v \mapsto (u \mapsto H(u, v))$. The form $H$ is non-degenerate if and only if $\varphi_{\mathcal{L}}$ is an isogeny. Moreover in this case, the degree of $\phi_{\mathcal{L}}$ equals $\det(c_1(\mathcal{L})$.

It turns out that the form $H$ is positive definite if and only if $\mathcal{L}$ is ample and hence some power $\mathcal{L}^n$ gives rise to an embedding of $X$ into the projective space $\mathbb{P}(H^0(X, \mathcal{L}^n)^\wedge)$. By Chow's theorem any closed analytic subspace of projective space admits the structure of an algebraic variety. Conversely, it can be shown that any abelian variety over $\mathbb{C}$ admits a polarization and we have the following theorem.

**Theorem 1.** *A complex torus $X$ admits the structure of a projective algebraic variety if and only if it is polarizable.*

In view of this theorem, the theory of abelian varieties over $\mathbb{C}$ is largely reduced to linear algebra.

**Definition 2.** (a) *We define the category* PHS *of polarized Hodge structures as follows. The objects are triples $(\Lambda, \langle, \rangle, h)$ where $\Lambda$ is a free $\mathbb{Z}$-module of rank $2g$ for some $g$, $\langle, \rangle : \Lambda \times \Lambda \to \mathbb{Z}$ is a non-degenerate alternating form and $h$ is an $\mathbb{R}$-algebra homomorphism $\mathbb{C} \to \operatorname{End}_{\mathbb{R}}(\Lambda \otimes \mathbb{R})$ such that*

(1) *$\langle h(i)u, h(i)v\rangle = \langle u, v\rangle$ for all $u, v \in \Lambda_{\mathbb{R}}$*

(2) *the bilinear symmetric form on $\Lambda_{\mathbb{R}}$, $(u, v) \mapsto \langle u, h(i)v\rangle$ is positive definite.*

*The morphisms from $(\Lambda, \langle, \rangle, h)$ to $(\Lambda', \langle, \rangle', h')$ consist of $\mathbb{Z}$-module morphisms $f : \Lambda \to \Lambda'$ such that $f \circ h = h'$ and $\langle, \rangle' \circ f = \langle, \rangle$.*

(b) *We define the category* $\mathbb{Q}$-PHS *of rational polarized Hodge structures as follows: objects are triples $(V, \langle, \rangle, h)$ where $V$ is a $\mathbb{Q}$-vector space of dimension $2g$, $\langle, \rangle : V \times V \to \mathbb{Q}$ is a non-degenerate alternating form and $h$ is an $\mathbb{R}$-algebra homomorphism $\mathbb{C} \to \operatorname{End}_{\mathbb{R}}(V \otimes \mathbb{R})$ such that (1) and (2) above hold for the extension of $\langle, \rangle$ to $V_{\mathbb{R}}$. The morphisms from an object $(V, \langle, \rangle, h)$ to an object $(V', \langle, \rangle', h')$ are $\mathbb{Q}$-vector space morphisms $f : V \to V'$ such that $f \circ h = h'$ and $\langle, \rangle' \circ f = a\langle, \rangle$ for some $a \in \mathbb{Q}$.*

We similarly define the category PAV of polarized abelian varieties over $\mathbb{C}$ to consist of pairs $(A, H)$ where $A$ is an abelian variety over $\mathbb{C}$ and $H$ is a polarization on $A$. Morphisms from $(A, H)$ to $(A', H')$ are morphisms $A \to A'$ such that $f^*H' = H$. They form a group $\operatorname{Mor}_{\mathrm{PAV}}((A, H), (A', H'))$.

The category PAV /(isog) of polarized abelian varieties up to isogeny over $\mathbb{C}$ consists of pairs $(A, H)$ where $A$ is an abelian variety and $H$ is a polarization on $A$. The set of morphisms from $(A, H)$ to $(A', H')$ is defined be $\operatorname{Mor}_{\mathrm{PAV}}((A, H), (A', H') \otimes_{\mathbb{Z}} \mathbb{Q}$.

There is a functor PAV → PHS, sending $(A, H)$ to $(H_1(A, \mathbb{Z}), \operatorname{Im} H_{|_{H_1(A,\mathbb{Z})}}, h)$ where $h$ comes from the natural identification of $\operatorname{Lie} A$ with $H_1(A, \mathbb{R})$ and the complex structure on $\operatorname{Lie} A$. This is an equivalence of categories (there is a natural functor going in the opposite direction).

Similarly there is a functor PAV /(isog) → $\mathbb{Q}$-PHS sending an object $(A, H)$ to the triple $(H_1(A, \mathbb{Q}), \operatorname{Im} H_{|_{H_1(A,\mathbb{Q})}}, h)$ where $h$ arises as above. This functor can easily be checked to be fully faithful and essentially surjective, and hence is an equivalence of categories.

## 2. Endomorphisms of Polarized Abelian Varieties over $\mathbb{C}$

Let $(A, H)$ be a polarized abelian variety over $\mathbb{C}$. Let

$$B = \mathrm{End}(A)^\circ := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then $B$ is a finite dimensional semisimple $\mathbb{Q}$-algebra which is a division algebra if and only if $A$ is simple (that is, not isogenous to a product of two abelian varieties.). Let $V = H_1(A, \mathbb{Q})$ and $\langle , \rangle : V \times V \to \mathbb{Q}$ be the alternating non-degenerate form $\mathrm{Im}\, H_{|_V}$. Then $B$ acts faithfully on $V$ and the form $\langle , \rangle$ gives rise to an (anti-)involution $* : B \to B$ defined by

$$\langle bu, v\rangle = \langle u, b^* v\rangle$$

for all $u, v \in V$. This is called the *Rosati involution.* This involution can also be described as follows. The polarization $H$ arises from $c_1(\mathcal{L})$ for some line bundle $\mathcal{L}$ on $A$ such that

$$\varphi_{\mathcal{L}} : A \to \hat{A}$$

is an isogeny. The isogeny $\varphi_{\mathcal{L}}$ has an inverse $\varphi_{\mathcal{L}}^{-1}$ in $\mathrm{Hom}(\hat{A}, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and we have

$$b^* = \varphi_{\mathcal{L}}^{-1} \hat{b} \varphi_{\mathcal{L}}$$

where $\hat{b} \in \mathrm{End}(\hat{A})^\circ$ is the dual endomorphism. Since the symmetric bilinear form $V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}; (u, v) \mapsto \langle u, h(i)v\rangle$ is positive definite and since the action of $h(i)$ on $V_{\mathbb{R}}$ commutes with that of $B$, it can be shown that,

$$\mathrm{tr}_{B/\mathbb{Q}}(bb^*) > 0$$

for all $b \in B$. An involution with this property is called *positive.*

Let $B$ be a division algebra, finite dimensional over $\mathbb{Q}$, with positive involution $*$. Let $F$ denote the center of $B$ and let $F^+$ denote the fixed field of $*$ restricted to $F$. We say that $*$ is *of the first kind* if $F^+ = F$ and $*$ is *of the second kind* otherwise. It is not hard to show, using weak approximation, that the positivity of $*$ implies

- if $*$ is of the first kind, then $F = F^+$ is totally real
- if $*$ is of the second kind, $F^+$ is totally real and $F$ is a totally imaginary quadratic extension of $F^+$ with $*_{|_F}$ equal to complex conjugation.

Albert has classified such pairs $(B, *)$. Here is the result in the case where $*$ is of the first kind. Firstly, if $B$ is a quaternion algebra over a totally real number field $F^+$, then $B$ carries a canonical involution $\overline{b} = \mathrm{tr}_{B/F^+}(b) - b$ (which may or may not be positive).

**Theorem 3.** *Let $B$ be a division algebra, finite dimensional over $\mathbb{Q}$, with a positive involution of the first kind $*$. Let $F^+$ be the center of $B$. Then one of the following holds*

(i) *$B = F^+$ and $x^* = x$ for all $x \in F^+$.*

(ii) *$B$ is a quaternion algebra over $F^+$ and for every embedding $\tau : F^+ \to \mathbb{R}$ there is an isomorphism*

$$\phi_\tau : B \otimes_{F^+, \tau} \mathbb{R} \to M_2(\mathbb{R}).$$

*There exists an element $a \in B$ with $a^2 \in F^+$ totally negative such that $b^* = a^{-1} \overline{b} a$.*

(iii) *$B$ is a quaternion algebra over $F^+$ and for every embedding $\tau : F^+ \to \mathbb{R}$ there is an isomorphism*

$$\phi_\tau : B \otimes_{F^+, \tau} \mathbb{R} \to \mathbb{H}.$$

*The involution $*$ is given by $b^* = \overline{b}$ for all $b \in B$.*

Here is the result in the case where $*$ is of the second kind:

**Theorem 4.** *Let $B$ be a division algebra of finite dimension over $\mathbb{Q}$ with center $F$, a totally complex quadratic extension of a totally real number field $F^+$. Suppose $B$ admits an involution of the second kind on $B$. Then $B$ admits a positive involution of the second kind. Moreover, if $*$ is a positive involution of the second kind on $B$, then for each embedding $\sigma : F \to \mathbb{C}$ there is an isomorphism*

$$\phi_\sigma : B \otimes_{F,\sigma} \mathbb{C} \to M_d(\mathbb{C})$$

*such that $\phi_\sigma(b^*) = {}^t\overline{\phi_\sigma(b)}$. Any other positive involution of the second kind $b \to b'$ on $B$ is given by $b' = ab^*a^{-1}$ with $a \in B$, $a^* = a$ and such that $\phi_\sigma(a \otimes 1)$ is a positive definite Hermitian matrix for all $\sigma$ as above.*

## 3. Unitary Shimura Varieties - definition of the group

In summary, a polarized abelian variety $(A, H)$ over $\mathbb{C}$ gives rise to a tuple $(B, *, V, \langle , \rangle, h)$ where $B$ is a finite dimensional semi-simple $\mathbb{Q}$ algebra, $*$ is a positive involution on $B$, $V$ is a faithful, finitely generated left $B$-module, $\langle , \rangle : V \times V \to \mathbb{Q}$ is a non-degenerate alternating form and $h : \mathbb{C} \to \mathrm{End}_{B_\mathbb{R}}(V_\mathbb{R})$ is an $\mathbb{R}$-algebra homomorphism. This tuple satisfies the following properties:

1. We have $\langle h(i)u, h(i)v\rangle = \langle u, v\rangle$ for all $u, v \in V_\mathbb{R}$. Equivalently, $\langle h(z)u, v\rangle = \langle u, h(\overline{z})v\rangle$ for all $u, v \in V_\mathbb{R}$ and $z \in \mathbb{C}$.
2. The symmetric bilinear form $V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{R}; (u, v) \mapsto \langle u, h(i)v\rangle$ is positive definite.
3. For all $b \in B$ and $u, v \in V$ we have $\langle bu, v\rangle = \langle u, b^*v\rangle$.

We call an alternating pairing $\langle , \rangle : V \times V \to \mathbb{Q}$ which satisfies (3) a $*$-Hermitian form.

Our goal is to fix a tuple $(B, *, V, \langle , \rangle)$ satisfying (3) with, in addition, $B$ being a division algebra, and to consider the 'space' of 'complex structures' $h : \mathbb{C} \to \mathrm{End}_{B_\mathbb{R}}(V_\mathbb{R})$ satisfying (1) and (2). Equivalently we want to consider the space of isogeny classes polarized abelian varieties $(A, H)$ over $\mathbb{C}$ with endomorphisms by $B$ (that is $B \to \mathrm{End}(A)^\circ$) such that $H_1(A, \mathbb{Q})$, together with the Rosati involution is isomorphic as to $(V, \langle , \rangle)$ as a skew-Hermitian $B$-module. This will be made explicit in Ana's talk. For now we will define an algebraic group $G$ over $\mathbb{Q}$, associated to the data $(B, *, V, \langle , \rangle)$, whose group of real points $G(\mathbb{R})$ will act transitively on the set of complex structures $h$.

The group $G$, and related groups $G_1$ and $G_0$ are defined as follows. Firstly, let

$$C = \mathrm{End}_B(V).$$

Then $C$ is a central simple $F$-algebra. The pairing $\langle , \rangle : V \times V \to \mathbb{Q}$ defines an involution of the second kind $\# : C \to C$ by requiring that

$$\langle cu, v\rangle = \langle u, c^\# v\rangle$$

for all $u, v \in V$ and $c \in C$. We let $G$ denote the reductive algebraic group over $\mathbb{Q}$ whose group of $R$ points for any $\mathbb{Q}$-algebra is given by

$$G(R) = \{(\lambda, g) \in R^\times \times (C \otimes_\mathbb{Q} R)^\times : gg^\# = \lambda \in R^\#\}.$$

This group admits a homomorphism $\nu : G \to \mathbb{G}_m$ which, on $R$-points, sends a pair $(\lambda, g)$ to $\lambda \in R^\times$. Let $G_1$ denote the kernel of $\nu$. We let $G_0$ denote the algebraic group over $F^+$ whose $R$-points for any $F^+$-algebra $R$ is the set

$$G_0(R) = \{g \in (C \otimes_{F^+} R)^\times : gg^\# = 1\}.$$

Moreover $G_1$ is the restriction of scalars of $G_0$ over $F^+$ to $\mathbb{Q}$.

Suppose now that $V = B$ as a $B$-module and let $n = \dim_F B$. Then there is an isomorphism

$$C = \mathrm{End}_B(V) \to B^{\mathrm{op}}$$

which sends $\varphi$ to $\varphi(1)$. Here $B^{\mathrm{op}}$ acts on $B$ via right multiplication. Note that

$$V_{\mathbb{R}} = V \otimes_{F^+} (F^+ \otimes_{\mathbb{Q}} \mathbb{R}) = \bigoplus_{\tau : F^+ \to \mathbb{R}} V \otimes_{F^+,\tau} \mathbb{R}$$

as a module over

$$(B \otimes_F B^{\mathrm{op}}) \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\tau : F^+ \to \mathbb{R}} B_\tau \otimes_{\mathbb{C}} B_\tau^{\mathrm{op}}$$

where $B_\tau = B \otimes_{\mathbb{F}^+,\tau} \mathbb{R}$ and, $B_\tau^{\mathrm{op}}$ is defined similarly. For each $\tau : F^+ \to \mathbb{R}$ choose an embedding $\widetilde{\tau} : F \to \mathbb{C}$ extending $\tau$. Then there is an isomorphism

$$\phi_{\widetilde{\tau}} : B_\tau = B \otimes_{F^+,\tau} \mathbb{R} = B \otimes_{F,\widetilde{\tau}} \mathbb{C} \to M_n(\mathbb{C})$$

under which $*$ corresponds to the standard involution $X \mapsto {}^t\overline{X}$. Let $V_\tau = V \otimes_{F^+,\tau} \mathbb{R}$ for each $\tau$ as above. The decomposition $V_{\mathbb{R}} = \oplus_\tau V_\tau$ is orthogonal with respect to the pairing $\langle , \rangle$. Let $\langle , \rangle_\tau$ denote the pairing restricted to $V_\tau \times V_\tau$. Let $\#_\tau$ denote the involution on $B_\tau^{\mathrm{op}}$ defined by $\#_\tau$. Then it follows that

$$G_1(\mathbb{R}) = \prod_{\tau : F^+ \to \mathbb{R}} \{ g \in (B_\tau^{\mathrm{op}})^\times : g g^{\#_\tau} = 1 \}.$$

The factor corresponding to $\tau$ is the group $G_0(F_\tau^+)$.

We now show that the group $G_1(\mathbb{R})$ is isomorphic to a product of unitary groups $U(p,q)$ for some pairs $(p,q)$. By definition $U(p,q)$ is the subgroup of $GL_n(\mathbb{C})$, where $n = p+q$, preserving the Hermitian form

$$\begin{aligned} \langle , \rangle_{(p,q)} : \mathbb{C}^n \times \mathbb{C}^n &\to \mathbb{C} \\ (v,w) &\mapsto \sum_{i=1}^{p} \overline{v_i} w_i - \sum_{i=p+1}^{n} \overline{v_i} w_i. \end{aligned}$$

Moreover, any non-degenerate Hermitian form on $\mathbb{C}^n$ is equivalent to $\langle , \rangle_{(p,q)}$ for a unique $(p,q)$.

The involution $\#$ on $B_{\mathbb{R}}^{\mathrm{op}}$ corresponds to a non-degenerate Hermitian form $\langle , \rangle_\#$ on the $\mathbb{C}$-dual of $\epsilon V_{\mathbb{R}}$, which we denote $W$. This form is defined up the usual equivalence together with multiplication by an element of $\mathbb{R}^\times$. Indeed the relation between the involution $\#$ and the form $\langle , \rangle_\#$ is that

$$\langle cu, v \rangle_\# = \langle u, c^\# v \rangle_\#.$$

Let $\langle , \rangle_{\#,\tau}$ denote the restriction of $\langle , \rangle_\#$ to $W_\tau$. The group $G_0(F_\tau^+)$ is the unitary group corresponding to this form. Thus if $(p_\tau, q_\tau)$ denotes the signature of $\langle , \rangle_{\#,\tau}$, then we have an isomorphism

$$G_0(F_\tau^+) \cong U(p_\tau, q_\tau).$$

Note that the tuple $(p_\tau, q_\tau)_\tau$ is only well defined up to the equivalence $(p_\tau, q_\tau)_\tau \sim (q_\tau, p_\tau)_\tau$.

From tomorrow onwards, we will specialize to the case where $n = 2$ and where

$$G_0(F_\tau^+) \cong U(1,1)$$

for a fixed choice of embedding $\tau : F^+ \to \mathbb{R}$ and

$$G_0(F_{\tau'}^+) \cong U(0,2)$$

for all $\tau' \neq \tau$.