FINAL PROJECT FOR COMPLEX MULTIPLICATION TUTORIAL 2005

Now we need to think about the final project for this tutorial. There are many good topics, but some of them require a good amount of background knowledge. I'll list some possible topics that might interest you, but you can choose your own one if you wish.

I'll mainly refer you to two user-friendly books: (Unfortunately the second one may not be available from libraries, but I can still make photocopies for you)

[Sil2] J. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, Springer GTM 151

[Cox] D. Cox, Primes of the Form $x^2 + ny^2$, Wiley Interscience

* About class numbers and j-invariants

1. Class number 1 problem - Which imaginary quadratic fields that have class number 1? (Remember, the class number is the order of the ideal class group) There are exactly nine of them. You may try to follow any version of proof (see [Cox] sec 7.D and sec 12.C), or work with a particular example: for instance, explain some interesting features of the number $\exp(\pi\sqrt{163})$ (See [Sil2] Ch 2, Ex 6.2.1)

2. Computation of many j-invariants for the imaginary quadratic fields with class number possibly bigger than 1. - First you may look at [Sil2] Ch 2, Ex 6.2.2. for explicit examples or [Cox] Sec 12.C and 12.D. After understanding these material, see if you can come up with wise algorithms for computation.

* Computing class equations or modular equations. (You've seen these from Teruyoshi's talk)

- This would be very interesting as well. (I'd be happy to see this.) You may begin by tackling Sec 13 of [Cox]. He has a good discussion there.

* Application to the diophantine equation $x^2 + ny^2 = p$

- In fact, this is one main theme underlying Cox's book, which can be seen from its title. The integral solution of $x^2 + ny^2 = p$ is related to the Hilbert class field of $Q(\sqrt{-n})$. This much is an application of class field theory. (See [Cox] p.110-115 and more) Don't worry; it's a special case and doesn't require much knowledge. You may want to understand this part first and then make connection with CM theory. One obvious connection is that you know how to find a generator of Hilbert class field of any imaginary quadratic field in a concrete way from CM theory. Try to include detailed recipes for the cases with some particular values of n.

* For theory-lovers

There are some useful references for advanced readers.

[La] S. Lang, Elliptic Functions, Addison-Wesley

[Sh] G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions, Princeton Univ Press

1. Translate the consequences in CM theory that we discussed so far into adelic language (as much as you can). You may add ingredients from [La], [Sh] or [Sil2].

2. Prove that the j-invariant is an algebraic integer at a CM-point, by algebraic method (use good reduction modulo p). The outline of the proof is given in [Sil2], Ch 2, Thm 6.4, p.148-151 and you can find more references there. You may need to import some results without giving proofs, but see how much self-contained you can make it.