UNITARY SHIMURA VARIETIES OVER $\mathbb{C}$ - PART 2

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Let B be a quaternion algebra over the field $F = EF^+$, where E is an imaginary quadratic extension of $\mathbb{Q}$ and F^+ is a totally real field of degree d. Let V be the $\mathbb{Q}$-module B and let $C = \mathrm{End}_B(V)$. C can be identified with B^{op} via $\varphi \mapsto \varphi(1)$. We are also given a positive involution of the second kind on B denoted by $*$.

Recall that in David's talk the unitary group G was defined to be the similitude group preserving a certain alternating $*$-Hermitian pairing $(\ ,\ )$ on V, such that for any $\mathbb{Q}$-algebra R

$$G(R) = \{(\lambda, g) \in R^\times \times (C \otimes_{\mathbb{Q}} R)^\times | gg^\bullet = \lambda\}.$$

In the above definition, $g \mapsto g^\bullet$ is the involution on C defined by the condition

$$\langle gv, w\rangle = \langle v, g^\bullet w\rangle.$$

We can choose our Hermitian pairing such that

$$G(\mathbb{R}) = G(U(1,1) \times U(0,2)^{d-1}),$$

where the first factor corresponds to a chosen embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and the other $d-1$ factors correspond to the embeddings $\tau' : F^+ \hookrightarrow \mathbb{R}$ with $\tau' \neq \tau$.

Moreover, if R is an algebra over E, then $G(R)$ can be identified with the subset of

$$(C \otimes_{\mathbb{Q}} R) = (C \otimes_E R) \times (C \otimes_{E,c} R)$$

given by pairs (g,h) with $gh^\bullet \in R^\times$. Then $G(R)$ is identified with the direct product $R^\times \times (C \otimes_E R)^\times$ via $(g,h) \to (gh^\bullet, g)$. This means that, over E, we can think of the algebraic group G as just $\mathbb{G}_m$ times the algebraic group defined by B^{op}.

Now let's extend $\tau : F^+ \hookrightarrow \mathbb{R}$ to an embedding $F \hookrightarrow \mathbb{C}$. We can define a unitary Shimura variety $X_{U,\tau}(\mathbb{C})$ associated to the unitary group G and show that it is a moduli space for polarized abelian varieties over $\mathbb{C}$ with an action of B and level structure. Let $\mathfrak{h}_\tau$ denote the set of $I \in C_{\mathbb{R}} = \mathrm{End}_{B_{\mathbb{R}}}(V_{\mathbb{R}})$ which satisfty

(1) $I^2 = -1$.
(2) $I^\bullet = -I$.
(3) The pairing (v, Iw) is a positive definite symmetric form on $V_{\mathbb{R}}$.
(4) If $\tau' : F \hookrightarrow \mathbb{C}$ coincides with τ on E then

$$\dim_{\mathbb{C}}(V \otimes_{F,\tau'} \mathbb{C})^{I=i}$$

is 2 if $\tau = \tau'$ and 0 otherwise.

The first three properties that elements of $\mathfrak{h}_\tau$ must satisfy parametrize $\mathbb{R}$-algebra homomorphisms $h : \mathbb{C} \to \mathrm{End}_{B_{\mathbb{R}}}(V_{\mathbb{R}})$ such that $h(\bar{z}) = h(z)^\bullet$ and such that $(v, h(i)w)$ is a positive definite symmetric form. This is easy to see: in order to describe such a homomorphism it suffices to specify $I = h(i)$.

The condition that $(v, h(i)w)$ be positive definite is equivalent to asking that the involution on $C_{\mathbb{R}}$ given by $c \mapsto h(i)^{-1}c^\bullet h(i)$ be positive. Any two positive definite Hermitian modules are isomorphic as Hermitian B-modules if and only if they are isomorphic as B-modules. Therefore, requiring that the involution $c \mapsto h(i)^{-1}c^{\bullet}h(i)$ be positive ensures that the isomorphism class of $V_{\mathbb{R}}$ as a Hermitian $B_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$-module only depends on the $B_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$-module structure. This structure is pinned down by the fourth property.

Any two homomorphisms h, h' such that $h(i), h'(i) \in \mathfrak{h}_{\tau}$ are conjugate under $G_1(\mathbb{R}) = \{g \in C_{\mathbb{R}} | gg^{\bullet} = 1\}$. Indeed, h and h' will define isomorphic Hermitian $B_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$-module structures, there is some $c \in C_{\mathbb{R}}^{\times}$ realizing that isomorphism, i.e. $h'(i) = c \cdot h(i) \cdot c^{-1}$. Moreover, c should map one Hermitian pairing to another, i.e. $(cv, h'(i)cw) = (v, h(i)w)$. We get $c^{\bullet}h'(i)c = c^{-1}h'(i)c$, which means that $c^{\bullet}c = 1$ and so $c \in G_1(\mathbb{R})$. Thus, we've shown that h and h' are conjugate under some $G_1(\mathbb{R})$.

Note that this is the same as saying that h and h' are conjugate under $G(\mathbb{R})^+ = \{g \in C_{\mathbb{R}} | gg^{\bullet} > 0\}$. Let us now fix some $I \in \mathfrak{h}_{\tau}$ (or equivalently a homomorphism h) and let U_{τ} denote its centralizer in $G(\mathbb{R})^+$. Let $U \subset G(\mathbb{A}^{\infty})$ be a sufficiently small open compact subgroup. We define the Shimura curve

$$X_{U,\tau}(\mathbb{C}) = G(\mathbb{Q})^+ \backslash (G(\mathbb{A}^{\infty})/U \times \mathfrak{h}_{\tau}) \simeq G(\mathbb{Q}) \backslash (G(\mathbb{A}^{\infty})/U \times G(\mathbb{R})/U_{\tau}).$$

This Shimura curve is a moduli space for polarized abelian varieties over $\mathbb{C}$. More precisely, $X_{U,\tau}(\mathbb{C})$ will parametrize quadruples $(A, \lambda, i, \bar{\eta})$, where

(1) A is an abelian variety over $\mathbb{C}$;
(2) $\lambda : A \to A^{\vee}$ is a polarization (an isogeny between A and its dual abelian variety $A^{\vee}$).
(3) $i : B \hookrightarrow \mathrm{End}^0 A$ is a homomorphism such that (A, i) is **compatible** and such that $\lambda \circ i(b) = i(b^*)^{\vee} \circ \lambda$ for all $b \in B$.
(4) $\bar{\eta}$ is a U-orbit of isomorphisms of skew-Hermitian B-modules $V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty} \simeq VA$, where $VA = H_1(A, \mathbb{A}^{\infty})$ is the Tate module of A. Note that the B-module structure of $H_1(A, \mathbb{A}^{\infty})$ is induced from i. The alternating pairing on $H_1(A, \mathbb{A}^{\infty})$, called the λ-Weil pairing, is obtained by composing the polarization
$$\lambda : A \to A^{\vee}$$
and the duality
$$VA \times VA^{\vee} \to \mathbb{A}^{\infty}(1).$$
The relation $\lambda \circ i(b) = i(b^*)^{\vee} \circ \lambda$ ensures that the alternating pairing is $*$-Hermitian. Notice that the λ-Weil pairing will take its values in the Tate twist
$$\mathbb{A}^{\infty}(1) = (\lim_{\leftarrow} \mu_n) \otimes_{\mathbb{Z}} \mathbb{Q},$$
where μ_n is the module of n-th roots of unity. When viewed as an alternating form with values in $\mathbb{A}^{\infty}$, the λ-Weil pairing is only well defined up to an automorphism of $\mathbb{A}^{\infty}$. Therefore, we require that the isomorphism η takes the pairing $\langle , \rangle$ on V to an $(\mathbb{A}^{\infty})^{\times}$-multiple of the λ-Weil pairing on VA.

Now let's explain what we mean by (A, i) being compatible.

Recall that the fixed homomorphism h gives an action of $\mathbb{C}$ on $V_{\mathbb{R}}$. The vector space $V_{\mathbb{C}}$ decomposes as $V_1 \oplus V_2$, where $h(i)$ acts as i on V_1 and as $-i$ on V_2. The imaginary quadratic field E acts on V_1 via $\tau : E \hookrightarrow \mathbb{C}$ or via the conjugate embedding $\tau^c : E \hookrightarrow \mathbb{C}$. This further decomposes V_1 into the subspaces V_1^+ and

V_2^-. If we denote $\pm j$ be the square roots of -1 in

$$F \otimes_{F^+,\tau} \mathbb{R} \simeq \mathbb{C},$$

then the eigenvalues of $I = h(i)$ acting on $V_\mathbb{R}$ are $\pm j$. The embedding $\tau|_E : E \hookrightarrow \mathbb{C}$ maps $j \mapsto i$ and we know that the $\dim(V \otimes_{F,\tau} \mathbb{C})^{I=i} = 2$. Notice that this is exactly the subspace V_1^+ where $h(i) = I = i$ and E acts via τ. The other direct summand V_1^- is the subspace where $h(i) = I = i$ and E acts via the embedding $\tau^c|_E : E \hookrightarrow \mathbb{C}$ which maps $-j \mapsto i$. We deduce that $\dim V_1^- = 4d - 2$.

The condition that (A, i) be compatible requires Lie$A$ to be isomorphic to V_1, so that Lie$A$ decomposes into $\mathrm{Lie}^+ A \oplus \mathrm{Lie}^- A$, where E acts via τ and τ^c respectively and where $\mathrm{Lie}^+ A$ has dimension 2 and $\mathrm{Lie}^- A$ has dimension $4d - 2$.

Now that we've explained the conditions the quadruples $(A, \lambda, i, \bar{\eta})$ must satisfies, we'll show that every such quadruple gives rise to an element of $G(\mathbb{A}^\infty)/U \times \mathfrak{h}_\tau$. Let $H = H_1(A, \mathbb{Q})$. The isomorphism η ensures that H and V are isomorphic as skew-Hermitian B-modules over any single finite place of $\mathbb{Q}$. Moreover, we find that $H_\mathbb{R} \simeq V_\mathbb{R}$ at least as B-modules. We know that $H_\mathbb{R} \simeq$ Lie$A$ as real vector spaces. The complex structure on Lie$A$ induces a complex structure on $H_\mathbb{R}$ and we get a decomposition $H_\mathbb{C} = H_1 \oplus H_2$. We also have $V_\mathbb{C} = V_1 \oplus V_2$. The compatibility condition tells us that $V_1 \simeq H_1$ as $B_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$-modules, and so $V_\mathbb{R} \simeq H_\mathbb{R}$ are also isomorphic as $B_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$-modules. We deduce that $V_\mathbb{R} \simeq H_\mathbb{R}$ are in fact isomorphic as skew-Hermitian $B_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$-modules. We let η_{dR} be a rigidification of $H_\mathbb{R}$. The complex structure on $H_\mathbb{R}$ is obtained by conjugating the fixed $h : \mathbb{C} \to \mathrm{End}_{B_\mathbb{R}}(V_\mathbb{R})$ by $\eta_{dR} \in G(\mathbb{R})$. Thus, the moduli space for the possible complex structures on $H_\mathbb{R}$ is exactly $\mathfrak{h}_\tau$.

We've seen that H and V are isomorphic as skew-Hermitian modules over every place of $\mathbb{Q}$. When B is a quaternion algebra over F (or more generally a $(2n)^2$-dimensional division algebra over F) the local-global principle holds so that H and V have to be isomorphic over $\mathbb{Q}$. Let us choose some isomorphism $\eta_\mathbb{Q} : H_1(A, \mathbb{Q}) \to V$. Composition with an automorphism of V in $G(\mathbb{Q})^+$ simply takes $\eta_\mathbb{Q}$ to another isomorphism $\eta'_\mathbb{Q}$. Then the map $(\eta_\mathbb{Q} \otimes \mathbb{A}^\infty) \circ \eta$

$$V \otimes \mathbb{A}^\infty \to H_1(A, \mathbb{A}^\infty) \simeq H_1(A, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{A}^\infty \to V \otimes \mathbb{A}^\infty$$

determines an automorphism of $V \otimes \mathbb{A}^\infty$ as a skew-Hermitian B-module, so an element of $G(\mathbb{A}^\infty)$. Since the isomorphism $\bar{\eta}$ is defined only as a U-orbit, we only get a well-defined element of $G(\mathbb{A}^\infty)/U$.

The tuple $(A, \lambda, i, \bar{\eta}, \eta_\mathbb{Q})$ determines an element of $G(\mathbb{A}^\infty)/U \times \mathfrak{h}_\tau$. Forgetting the isomorphism $\eta_\mathbb{Q}$ amounts to quotienting out by the left action of $G(\mathbb{Q})^+$. Thus, the quadruple $(A, \lambda, i, \bar{\eta})$ determines a point on

$$X_{U,\tau}(\mathbb{C}) = G(\mathbb{Q})^+ \backslash (G(\mathbb{A}^\infty)/U \times \mathfrak{h}_\tau).$$

The theory of abelian varieties over $\mathbb{C}$ can be used to see that any point on $X_{U,\tau}(\mathbb{C})$ comes from an abelian variety up to an isogeny.

Finally, we want to show that $\mathfrak{h}_\tau$ can be given a complex structure which identifies it with the unit disc subset of $\mathbb{P}(\mathbb{C})$. Let's identify $V_{\mathbb{R},\tau}$ (the subspace of $V_\mathbb{R}$ corresponding to the embedding τ) with $M_2(\mathbb{C})$.Let $I = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$, where j is a square root of -1 in $\mathbb{C}$. If $\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $\epsilon V \simeq \mathbb{C}^2$. Any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$

gives $I' = gIg^{-1} \in \mathfrak{h}_\tau$. I' acts on ϵV with eigenvalues $j, -j$ and eigenvectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$. Sending I' to the one-dimensional subspace of ϵV generated by $\begin{pmatrix} b \\ d \end{pmatrix}$, or equivalently to $(\frac{b}{d}, 1) \in \mathbb{P}(\mathbb{C})$. Thus, we get a map

$$\mathfrak{h}_\tau \to \mathbb{P}(\mathbb{C})$$

whose image is exactly the image of 0 in the unit disc under $z \mapsto \frac{az+b}{cz+d}$. This image is 0 only when $b = c = 0$, in which case the element g is in the centralizer of I. Therefore, we have an isomorphism of $\mathfrak{h}_\tau$ with the unit disc subset of $\mathbb{P}(\mathbb{C})$ and a canonical complex structure on $\mathfrak{h}_\tau$.