SURVEY ON NON-ABELIAN LUBIN-TATE THEORY

TERUYOSHI YOSHIDA

Abstract. We give a survey, for non-experts, of the non-abelian Lubin-Tate theory, a cohomological realization of the local Langlands correspondence over p-adic fields. As its proof at the moment (mostly given by the work of Harris-Taylor [HT]) requires global techniques using certain Shimura varieties, we will treat the arithmetic geometry of these Shimura varieties, including some of the recent developments.

Contents

1. Notation

The cardinality of a finite set X is denoted by $|X|$. For a ring A, we write $A^\times$ for its group of units. For a field F, we usually (implicitly) fix its algebraic closure $\overline{F}$ and regard any algebraic extension of F as a subfield of $\overline{F}$. For a finite extension F'/F, we denote the norm map by $N_{F'/F} : F'^\times \to F^\times$. We denote the maximal abelian extension of F in $\overline{F}$ by F^{ab}. For a perfect field F, we write $G_F := \mathrm{Gal}(\overline{F}/F)$ and $G_F^{\mathrm{ab}} := \mathrm{Gal}(F^{\mathrm{ab}}/F)$. For a positive integer n not divisible by $\mathrm{char}\, F$, the splitting field of $X^n - 1$ over F is denoted by $F(\boldsymbol{\mu}_n)$ (*cyclotomic extension*), which is an abelian extension whose Galois group naturally injects into $(\mathbb{Z}/(n))^\times$. We denote the set of roots of $X^n - 1$ by $\boldsymbol{\mu}_n$. We denote the finite field consisting of q elements by $\mathbb{F}_q$. For each $n \geq 1$, we have $\mathbb{F}_{q^n} = \mathbb{F}_q(\boldsymbol{\mu}_{q^n-1})$. The Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is isomorphic to $\widehat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/(n)$, the inverse limit with respect to the canonical surjection $\mathbb{Z}/(n') \to \mathbb{Z}/(n)$ for $n \mid n'$ (the *profinite completion* of $\mathbb{Z}$), by sending the q-th power *Frobenius map* $x \mapsto x^q$ to -1.

2. LOCAL CLASS FIELD THEORY AND LUBIN-TATE THEORY

2.1. **Local class field theory.** We are interested in the absolute Galois group G_K of a *local field* K, which means a finite extension of p-adic field $\mathbb{Q}_p$ in this article. Local class field theory describes the maximal abelian quotient G_K^{ab} of G_K.

2.1.1. *Local fields.* Fix a prime number p, and let K be a finite extension of the p-adic field $\mathbb{Q}_p$. It is a complete discrete valuation field with the ring of integers $\mathcal{O} := \mathcal{O}_K$ and the maximal ideal $\mathfrak{p}$, and let $k := \mathcal{O}/\mathfrak{p} \cong \mathbb{F}_q$ be its residue field, where q is a power of p. For any discrete valuation field (like K), a generator of its maximal ideal is called its *uniformizer.*

We are interested in understanding the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ of K. Let $K^{\mathrm{ur}} := \bigcup_{(p,N)=1} K(\boldsymbol{\mu}_N)$ be the *maximal unramified extension* of K. By Hensel's lemma, its Galois group $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ is canonically isomorphic to $\mathrm{Gal}(\overline{k}/k) \cong \widehat{\mathbb{Z}}$ via $\sigma \longmapsto (\sigma|_{\mathcal{O}} \bmod \mathfrak{p})$, and let $\mathrm{Frob}_K \in \mathrm{Gal}(K^{\mathrm{ur}}/K)$ be the element mapping to $(x \mapsto x^q)^{-1} \in \mathrm{Gal}(\overline{k}/k)$ (the *geometric Frobenius element*). Let

$$f : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(K^{\mathrm{ur}}/K) \xrightarrow{\cong} \mathrm{Gal}(\overline{k}/k) \cong \widehat{\mathbb{Z}}$$

be the composite map. We call $W_K := f^{-1}(\mathbb{Z})$ the *Weil group* of K, and $I_K := \mathrm{Ker}\, f$ the *inertia group* of K. We define a topology on W_K so that I_K (with its natural profinite topology) becomes an open subgroup of W_K. This topology on W_K is slightly stronger than the topology induced from G_K.

2.1.2. *Local class field theory.* Local class field theory describes $G_K^{\mathrm{ab}} := \mathrm{Gal}(K^{\mathrm{ab}}/K)$. Note that the image W_K^{ab} of W_K in G_K^{ab} is the maximal abelian quotient of W_K.

Theorem 2.1 (Local class field theory)**.** *There is a unique homomorphism* $\mathrm{Art}_K : K^\times \longrightarrow G_K^{\mathrm{ab}}$ *characterized by the following properties:*

(i) *For every uniformizer* ϖ *of* K*, we have* $\mathrm{Art}_K(\varpi)|_{K^{\mathrm{ur}}} = \mathrm{Frob}_K$.
(ii) *For every finite abelian extension* K'/K*, we have* $\mathrm{Art}_K(N_{K'/K}(x))|_{K'} = \mathrm{id}$ *for* $\forall x \in K'^\times$.

Moreover, (a) Art_K *gives an isomorphism* $K^\times \xrightarrow{\cong} W_K^{\mathrm{ab}}$*, and (b) for every finite extension* K'/K*, we have* $\mathrm{Res}_{K'/K} \circ \mathrm{Art}_{K'} = \mathrm{Art}_K \circ N_{K'/K}$*, where* $\mathrm{Res}_{K'/K} : G_{K'}^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ *is the restriction map (the* base change property*).*

There are several ways to prove this theorem, among which the one via Galois cohomology ([Se]) and the one via Lubin-Tate theory ([Iw],[Yo3]) are important. Here we will treat the Lubin-Tate theory.

2.2. **Lubin-Tate theory.** In most elementary terms, Lubin-Tate theory is an explicit construction of all ramified abelian extension of K as follows. Recall that $q := |k|$ is the cardinality of the residue field of K.

Theorem 2.2. *For a uniformizer ϖ of K^{ur}, let $f_\varpi(X) = \varpi X + X^q$, and let $f_\varpi^m := f_\varpi(\cdots f_\varpi(f_\varpi(X))\cdots)$ be its m-fold iterate of f_ϖ for $m \geq 1$. Then the splitting field K_m of f_ϖ^m over K^{ur} is abelian over K and independent of ϖ, and $K^{\mathrm{ab}} = \bigcup_{m \geq 0} K_m$.*

To compute the Galois group of these extensions K_m/K^{ur}, we re-interpret f_ϖ as a ϖ-multiplication map of a *formal $\mathcal{O}$-module* of height 1.

2.2.1. *Formal $\mathcal{O}$-modules.*

Definition 2.3. A *formal group* over a ring A is a formal power series of two variables $F(X, Y) \in A[[X, Y]]$ such that $F(X, Y) \equiv X + Y \pmod{\deg 2}$ and:

$$F(F(X, Y), Z) = F(X, F(Y, Z)), \qquad F(X, Y) = F(Y, X).$$

The basic examples are the *additive group* $\widehat{\mathbb{G}}_a(X, Y) := X + Y$ and the *multiplicative group* $\widehat{\mathbb{G}}_m(X, Y) := X + Y + XY$. For a formal group F, there is a unique $[-1]_F \in A[[X]]$ satisfying $F(X, [-1]_F(X)) = 0$. If we define $f +_F g := F(f, g)$ for $f, g \in (X) \subset A[[X]]$, then (X) becomes an abelian group with 0 as the identity and $[-1]_F \circ f$ as the inverse of f.

Definition 2.4. Let F, G be formal groups over A. A power series $f \in (X) \subset A[[X]]$ is called a *homomorphism* from F to G if it satisfies

$$f \circ F = G \circ f, \quad \text{i.e.} \quad f(F(X, Y)) = G(f(X), f(Y)),$$

and we write $f : F \to G$. Two homomorphisms compose via the composition of power series, with $f(X) = X$ as the identity $\mathrm{id} : F \to F$.

The set $\mathrm{Hom}(F, G)$ of all homomorphisms from F to G is an abelian group under $+_G$. Moreover, $\mathrm{End}(F) := \mathrm{Hom}(F, F)$ is a (not necessarily commutative) ring with $+_F$ as the addition and $\circ$ as the multiplication.

Definition 2.5. Let A be an $\mathcal{O}$-algebra. A *formal $\mathcal{O}$-module* over A is a pair $\Sigma = (F, [\cdot])$ of a formal group F over A and a ring homomorphism $[\cdot] : \mathcal{O} \to \mathrm{End}(F)$ such that $[a](X) \equiv aX \pmod{\deg 2}$ for $\forall a \in \mathcal{O}$.

For example, the *additive $\mathcal{O}$-module* $\widehat{\mathbb{G}}_a$ is the additive group $\widehat{\mathbb{G}}_a$ with $[a](X) := aX$ for $\forall a \in \mathcal{O}$. For a formal $\mathcal{O}$-module Σ over A and a morphism of $\mathcal{O}$-algebras $A \to B$, we can define a formal $\mathcal{O}$-module $\Sigma \otimes B$ over B in an obvious manner. A *homomorphism* of formal $\mathcal{O}$-modules is a homomorphism of formal groups commuting with $[a]$ for $\forall a \in \mathcal{O}$. Then the set $\mathrm{Hom}(\Sigma, \Sigma')$ of all homomorphisms from Σ to Σ' is an $\mathcal{O}$-module by $a \cdot f := [a] \circ f$ for $\forall a \in \mathcal{O}$, and $\mathrm{End}(\Sigma) := \mathrm{Hom}(\Sigma, \Sigma)$ is an $\mathcal{O}$-algebra, as before.

If $\Sigma = (F, [\cdot])$ is a formal $\mathcal{O}$-module over a complete local ring $(A, \mathfrak{m})$, then $\mathfrak{m}$ becomes an $\mathcal{O}$-module $\mathfrak{m}_\Sigma$ (different from the $\mathcal{O}$-module structure induced from that of A) by defining $x +_\Sigma y := F(x, y),\ a \cdot_\Sigma x := [a](x)$ for $\forall x, y \in \mathfrak{m}$ and $\forall a \in \mathcal{O}$.

A starting point of the theory of formal $\mathcal{O}$-module is:

Proposition 2.6. ([Dr]) *A formal $\mathcal{O}$-module over a field of characteristic 0 is always isomorphic to the additive $\mathcal{O}$-module $\widehat{\mathbb{G}}_a$. A formal $\mathcal{O}$-module over a separably closed field of characterstic p is, if not isomorphic to $\widehat{\mathbb{G}}_a$, classified up to isomorphism by the* height $n \in \mathbb{Z}_{>0}$, *defined as follows: Σ has height n if $[\mathfrak{p}] = (X^{q^n})$ as ideals in $A[[X]]$, where $[\mathfrak{p}]$ is the ideal generated by $\{[a](X) \mid a \in \mathfrak{p}\}$ (we have $[\mathfrak{p}] = \big([\varpi]\big)$ for any uniformizer ϖ of K).*

For example, the multiplicative group $\widehat{\mathbb{G}}_m$ over $\overline{\mathbb{F}}_p$ naturally is a formal $\mathbb{Z}_p$-module, and its height is 1 because $[p](X) = (1+X)^p - 1 \equiv X^p \pmod{p}$. For a local ring A with a residue field κ of characteristic p, a formal $\mathcal{O}$-module Σ over A is said to have *height n* if $\Sigma \otimes \kappa^{\mathrm{sep}}$ has height n.

2.2.2. *Lubin-Tate groups.* The term *Lubin-Tate group* usually refers to a formal $\mathcal{O}$-module of height 1 over a complete unramified extension $\mathcal{O}_L$ of $\mathcal{O}$, i.e. a finite unramified extension or the completion of an infinite unramified extension. In particular, the completion $\widehat{K} := \widehat{K}^{\mathrm{ur}}$ of K^{ur} contains every complete unramified extension of K. We write $\widehat{\mathcal{O}} := \mathcal{O}_{\widehat{K}}$. A uniformizer of K is a uniformizer of L but not vice versa.

Proposition 2.7. *Let L be a complete unramified extension of K. For every power series $f \in \mathcal{O}_L[[X]]$ satisfying (1) $f(X) \equiv \varpi X \pmod{\deg 2}$ for some uniformizer ϖ of L, and (2) $f(X) \equiv X^q \pmod{\mathfrak{p}}$, there exists a Lubin-Tate group Σ_f over $\mathcal{O}_L$ such that $[\varpi] = f$. If L/K is finite, the isomorphism class of $\Sigma_f/\mathcal{O}_L$ depends only on $N_{L/K}(\varpi)$. If $L = \widehat{K}$, these $\Sigma_f/\widehat{\mathcal{O}}$ are all isomorphic to each other.*

This allows us to compute the Galois action on the extensions K_m that we defined in the beginning of this section.

Proposition 2.8. *Let ϖ be a uniformizer of K^{ur}, and construct the extension K_m as in Theorem 2.2. Then the previous proposition gives $\Sigma_{f_\varpi}/\mathcal{O}_L$, where L is a finite unramified extension containing ϖ. The set of all roots of $f_\varpi^m = [\varpi^m]$ is a free $\mathcal{O}/\mathfrak{p}^m$-module of rank 1 by the $\mathcal{O}$-action $[\cdot]$ of Σ_{f_ϖ}, and there is an isomorphism:*

$$(\mathcal{O}/\mathfrak{p}^m)^\times \ni u \longmapsto (\alpha \mapsto [u](\alpha)) \in \mathrm{Gal}(K_m/K^{\mathrm{ur}})$$

where α is any root of $f_\varpi^m/f_\varpi^{m-1}$. Passing to the limit with respect to m, we recover $\mathrm{Art}_K|_{\mathcal{O}^\times} : \mathcal{O}^\times \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ur}})$.

Precisely speaking, if we define $K^{\mathrm{LT}} := \bigcup_m K_m$, then we can directly prove Theorem 2.1 with K^{ab} replaced by K^{LT} (see [Yo3]), and then we can prove $K^{\mathrm{LT}} = K^{\mathrm{ab}}$ (the *local Kronecker-Weber theorem*) by some ramification theory, to complete the proof of Theorem 2.1. The difficult part of proving Theorem 2.1 for K^{LT} is to establish the base change property – this can be done using the computation of the norm groups by Coleman's norm operator [Co].

3. Local Langlands correspondence and non-abelian Lubin-Tate theory

3.1. **Local Langlands correspondence.** The Local Langlands correspondence generalizes the local class field theory to capture all of G_K, not only G_K^{ab}. By $\mathrm{Art}_K : K^\times \cong W_K^{\mathrm{ab}}$, we can formulate the local class field theory as giving a bijection:

$$\{\text{ characters of } W_K\} \ni \chi \longmapsto \chi \circ \mathrm{Art}_K \in \{\text{ characters of } K^\times\}.$$

The *local Langlands correspondence* roughly extends the above bijection to a bijection

$$\{\ n\text{-dimensional representations of } W_K\} \longleftrightarrow \{\text{ representations of } GL_n(K)\},$$

where $\{\ \}$ denotes the isomorphism classes of representations. We need to define the both sides of the bijection carefully.

3.1.1. *Weil-Deligne representations.* First we start from the representations of W_K. As W_K is a locally profinite group, i.e. has an open subgroup which is profinite, it is helpful to consider continuous representations on vector spaces over $\overline{\mathbb{Q}}_\ell$ for some prime ℓ. When $\ell \neq p$, the pro-ℓ part of W_K has a simple structure. Recall that we had $\varphi : W_K/I_K \cong \mathbb{Z}$. The pro-$p$ part P_K of the profinite group I_K is called the *wild inertia group* of K, and the quotient I_K/P_K is called the *tame inertia group* of K. Representations of W_K or G_K which is trivial on P_K (resp. I_K) is called *tamely ramified* (resp. *unramified*). A basic example of an unramified representation is the *unramified cyclotomic character* $\chi : W_K \ni \sigma \longmapsto q^{-\varphi(\sigma)} \in \mathbb{Q}^\times$.

We have an isomorphism:

$$t : I_K \ni \sigma \overset{\cong}{\longmapsto} \left(\frac{\sigma(\varpi^{1/N})}{\varpi^{1/N}}\right) \in \varprojlim_{(p,N)=1} \boldsymbol{\mu}_N \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1),$$

where ϖ is a uniformizer of K but t does not depend on the choice of ϖ, and $\mathbb{Z}_\ell(1) := \varprojlim_m \boldsymbol{\mu}_{\ell^m}$ is the *ℓ-adic Tate twist.* For a prime $\ell \neq p$, the projection to $\mathbb{Z}_\ell(1)$ is called the *ℓ-adic tame character* $t_\ell : I_K \longrightarrow \mathbb{Z}_\ell(1)$. This is the pro-$\ell$ part of I_K.

Now $\exp \circ t_\ell : I_K \longrightarrow \mathbb{Z}_\ell^\times$ does not extend to a continuous character $W_K \to \overline{\mathbb{Q}}_\ell^\times$ (consider the action of conjugation by Frob_K). But in the higher dimensional case, $\sigma \longmapsto \exp\bigl(t_\ell(\sigma)N\bigr)$ for a nilpotent matrix N can be extended to a continuous representation $W_K \to GL_n(\overline{\mathbb{Q}}_\ell)$. In fact this is the only way to capture the pro-ℓ part in I_K with such representations. This allows us to define an ℓ-independent notion corresponding to ℓ-adic continuous representations of W_K.

Definition 3.1. A *Weil-Deligne (WD-) representation* of W_K over a field Ω is a triple (r, N, V) where V is a finite dimensional vector space over Ω, $r : W_K \to GL(V)$ is a representation such that $r|_{I_K}$ factors through a finite quotient of I_K, and $N \in \mathrm{End}(V)$ such that $r(\sigma)N = \chi(\sigma)\varphi(\sigma)N$ for $\forall \sigma \in W_K$. We say (r, N, V) is *Frobenius semisimple* if r is semisimple. We can define the *Frobenius semisimplification* of (r, N, V) by $(r, N, V)^{F-\mathrm{ss}} := (r^{\mathrm{ss}}, N, V)$.

Proposition 3.2. *For all prime $\ell \neq p$, we have a bijection:*

$$\begin{aligned} &\{\ n\text{-dimensional WD-representations of } W_K \text{ over } \overline{\mathbb{Q}}_\ell\ \} \\ &\longleftrightarrow \{\ \text{continuous representations } W_K \to GL_n(\overline{\mathbb{Q}}_\ell)\}, \end{aligned}$$

given by $(r, N, V) \longmapsto \big(\sigma \mapsto r(\sigma)\exp(t_\ell(\sigma)N) \in GL(V)\big)$. *We denote the inverse of this map by* $\rho \mapsto \mathrm{WD}(\rho)$.

Any field isomorphism $\iota : \Omega \cong \Omega'$ gives a bijection between the isomorphism classes of WD-representations over Ω and those over Ω' by $(r, N, V) \mapsto \big(\iota \circ r,\ \iota(N),\ V \otimes_\iota \Omega'\big)$ where we write $\iota : \mathrm{End}(V) \cong \mathrm{End}(V \otimes_\iota \Omega')$. This can be applied to $\overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_{\ell'} \cong \mathbb{C}$.

A WD-representation (r, N) is called *unramified* if $N = 0$ and $r|_{I_K} = 1$. Frobenius semisimple unramified WD-representations are classified by the unordered set of eigenvalues $\{\alpha_1, \dots, \alpha_n\} \in (\Omega^\times)^n/S_n$ of Frob_K. A basic example of n-dimensional Frobenius semisimple WD-representation with non-trivial N is given by:

$$\mathrm{Sp}_n := (r,\ N,\ \langle e_1, \dots, e_n\rangle), \quad r(\sigma)e_i := \chi(\sigma)^{i-1}e_i, \quad N(e_i) = e_{i+1}\ (e_{n+1} = 0).$$

Proposition 3.3. *Every Frobenius semisimple WD-representation of W_K is isomorphic to a direct sum of indecomposable ones, which are always of the form* $\mathrm{Sp}_n(r) := \mathrm{Sp}_n \otimes (r, 0, V)$ *for an irreducible r.*

3.1.2. *Irreducible smooth respresentions of GL_n.* Now the representations of $GL_n(K)$. This is also a locally profinite group, but we are interested in its *smooth* representations, which are "algebraic" in some sense. Contrary to the representations of W_K, interesting representations of $GL_n(K)$ are almost always infinite dimensional.

Definition 3.4. Let V be a vector space over a field Ω, and (π, V) be a representation $\pi : G \longrightarrow GL(V)$ of a locally profinite group G. An element $v \in V$ is called *smooth* if there is a open compact subgroup $U \subset G$ such that $v \in V^U := \{v \in V \mid gv = v\ (\forall g \in U)\}$ (i.e. v is a *U-fixed vector*), and (π, V) is called *smooth* if $\forall v \in V$ are smooth.

We explain why these representations are "algebraic". It is known that every irreducible smooth representation (π, V) of $G := GL_n(K)$ is *admissible*, i.e. for every open compact subgroup $U \subset G$, the subspace V^U is finite dimensional. For each U, we define the *Hecke algebra* $\Omega[U\backslash G/U]$ as follows. As a vector space over Ω, it is generated by the set of all double U-cosets in G. If we consider this space as a space of compactly supported U-bi-invariant function $G \longrightarrow \Omega$, then the multiplication is defined by the convolution product, which makes $\Omega[U\backslash G/U]$ a (usually non-commutative) finitely generated Ω-algebra. Coming back to an irreducible admissible representation (π, V), the subspace V^U is naturally a module over $\Omega[U\backslash G/U]$ by:

$$[UgU](v) := \sum_h hv \quad (\forall v \in V^U) \quad \text{if} \quad UgU = \coprod_h hU,$$

and if $V^U \neq 0$ then we can recover (π, V) from this module V^U.

For example, if $V^U \neq 0$ for a maximal compact subgroup $U := GL_n(\mathcal{O})$, then we call V an unramified spherical representation. These correspond to irreducible finitely generated modules over $\Omega[U\backslash G/U]$, which turns out to be a commutative Ω-algebra by the *Satake isomorphism*:

$$\Omega\big[GL_n(\mathcal{O})\backslash GL_n(K)/GL_n(\mathcal{O})\big] \cong \Omega[T_1^{\pm}, \dots, T_n^{\pm}]^{S_n},$$

where S_n is the symmetric group of n letters operating on $T_1, \dots, T_n$. Hence, for an algebraically closed Ω, all irreducible modules over this Hecke algebra are one-dimensional, and classified by the unordered set $\{\alpha_1, \dots, \alpha_n\} \in (\Omega^\times)^n/S_n$ of eigenvalues of $T_1, \dots, T_n$ (the *Satake parameters*).

3.1.3. *Local Langlands correspondence.*

Theorem 3.5 (Local Langlands correspondence, [HT], [He1])**.** *There is a bijection*

$$\{ \textit{ Frobenius semisimple } n\textit{-dimensional WD-representations of } W_K \textit{ over } \mathbb{C} \}$$
$$\ni \mathcal{L}(\pi) \longleftrightarrow \pi \in \{ \textit{ irreducible smooth representations of } GL_n(K) \textit{ over } \mathbb{C} \},$$

uniquely characterized by a certain set of properties (see e.g. [He2]*).*

Here we explain some of the special cases of this correspondence.

(i) When $n = 1$, then WD-representations are just characters of W_K^{ab}, and we have $\mathcal{L}(\pi) \circ \mathrm{Art}_K = \pi$.
(ii) Unramified spherical representation π correspond to unramified WD-representation $\mathcal{L}(\pi)$ whose set of Frobenius eigenvalues $\{\alpha_1, \dots, \alpha_n\} \in (\mathbb{C}^\times)^n/S_n$ is equal to the Satake parameters of π.

3.2. **Non-abelian Lubin-Tate theory.** Now we turn to the non-abelian Lubin-Tate theory, an explicit cohomological realization of the local Langlands correspondence. There are two realizations (see [Ca]), the Lubin-Tate space version and the Drinfeld symmetric space version. We only treat the former in this survey – the equivalence of two theories are shown in [Fa], [FGL].

3.2.1. *Deformation ring of formal modules.* Just as Lubin-Tate groups (formal $\mathcal{O}$-modules of height 1) described the local class field theory (local Langlands correspondence for GL_1), the theory of formal $\mathcal{O}$-modules of height n is used for the GL_n-theory.

Recall from Proposition 2.6 that there is a unique formal $\mathcal{O}$-module Σ_n of height n over $\overline{\mathbb{F}}_q$. Proposition 2.7 says that Σ_1 has a unique deformation to $\widehat{\mathcal{O}}$. We fix an isomorphism $\widehat{\mathcal{O}}/\mathfrak{p}\widehat{\mathcal{O}} \cong \overline{\mathbb{F}}_q$ and identify them.

Let $\mathscr{C}$ be the category of complete noetherian local $\widehat{\mathcal{O}}$-algebra $(A, \mathfrak{m})$ such that the structure homomorphism $\widehat{\mathcal{O}} \to A$ gives the isomorphism of residue fields $\overline{\mathbb{F}}_q \cong \overline{A} := A/\mathfrak{m}$. Consider the deformation functor $\mathscr{F}_0 : \mathscr{C} \to (\mathrm{Sets})$ sending $(A, \mathfrak{m})$ to the set of isomorphism classes of pairs (Σ, i), where Σ is a formal $\mathcal{O}$-module of height n over A (see the definition below Proposition 2.6) and $i : \Sigma_n \otimes_{\overline{\mathbb{F}}_q} \overline{A} \xrightarrow{\cong} \Sigma \otimes_A \overline{A}$.

Theorem 3.6. ([LT2], [Dr], [GH]) *The functor* $\mathscr{F}_0$ *is represented by an* $R_0 \in \mathscr{C}$, *which is isomorphic to* $\widehat{\mathcal{O}}[[T_1, \dots, T_{n-1}]]$. *Moreover, for a uniformizer* ϖ *of* K, *we can choose the coordinates* $T_1, \dots, T_n$ *so that the* ϖ*-multiplication on the universal formal* $\mathcal{O}$*-module* $\widetilde{\Sigma}$ *on* R_0 *is given by:*

$$[\varpi](X) \equiv \varpi X + \sum_{i=1}^{n-1} T_i X^{q^i} + X^{q^n} \pmod{\deg q^n + 1}.$$

It is known that $\mathcal{D} := \mathrm{End}(\Sigma_n)$ is the maximal order of $D := \mathrm{End}(\Sigma_n) \otimes_{\mathcal{O}} K$, which is a division algebra over K of Hasse invariant $1/n$ (see [Dr]). One way to see the generators of $\mathcal{D}$ is as follows. For a uniformizer ϖ of K, we can construct a formal $\mathcal{O}$-module Σ over $\mathcal{O}$ of height n (i.e. $\Sigma \otimes \overline{\mathbb{F}}_q \cong \Sigma_n$) such that $[\varpi](X) = \varpi X + X^{q^n}$. Then $\Sigma \otimes \mathbb{F}_q$ has the geometric Frobenius endomorphism $\Pi := X^q$, satisfying $\Pi^n = [\varpi]$. Moreover, for the unramified extension L/K of degree n, the $\Sigma \otimes \mathcal{O}_L$ becomes an $\mathcal{O}_L$-Lubin-Tate group, such that $[\zeta](X) = \zeta X$ for $\forall \zeta \in \boldsymbol{\mu}_{q^n-1} \subset \mathcal{O}_L^\times$ (see that it commutes with $[\varpi]$). Therefore $\mathrm{End}(\Sigma \otimes \mathbb{F}_{q^n})$ contains $\mathcal{O}_L[\Pi]$, a maximal order in a division algebra of dimension n^2 over K, whose Hasse invariant is $1/n$ because $\Pi \circ [\zeta] = [\zeta^q] \circ \Pi$.

Therefore the functor $\mathscr{F}_0$ has a natural right action of $\mathcal{D}^\times = \mathrm{Aut}(\Sigma_n)$ by $[d] : (\Sigma, i) \mapsto (\Sigma, i \circ d)$ for $\forall d \in \mathcal{D}^\times$, hence R_0 has the corresponding left $\mathcal{D}^\times$-action. Note that this action factors through $\mathcal{D}^\times/\mathcal{O}^\times$, because $[a] : (\Sigma, i) \xrightarrow{\cong} (\Sigma, i \circ [a])$ for $\forall a \in \mathcal{O}^\times$.

For an $m \geq 1$ and a formal $\mathcal{O}$-module $\Sigma = (F, [\cdot])$ over $(A, \mathfrak{m}) \in \mathscr{C}$, a *Drinfeld level* $\mathfrak{p}^m$*-structure* on Σ is an $\mathcal{O}$-homomorphism

$$\alpha : (\mathfrak{p}^{-m}/\mathcal{O})^n \longrightarrow \mathfrak{m}_\Sigma,$$

where $\mathfrak{m}_\Sigma$ is defined below Definition 2.5, such that:

$$[\mathfrak{p}] = \Big(\prod_{x \in (\mathfrak{p}^{-1}/\mathcal{O})^n} \big(X - \alpha(x)\big) \Big),$$

as ideals in $A[[X]]$. Then we define a functor $\mathscr{F}_m : \mathscr{C} \to (\mathrm{Sets})$ by sending $(A, \mathfrak{m})$ to the set of isomorphism classes of triples (Σ, i, α), where (Σ, i) is as in $\mathscr{F}_0$ and α is the Drinfeld level $\mathfrak{p}^m$-structure.

Theorem 3.7. ([Dr]) *The functor* $\mathscr{F}_m$ *is represented by an* $R_m \in \mathscr{C}$, *which is finite flat* R_0*-algebra by the natural forgetting morphism of functors* $\mathscr{F}_m \to \mathscr{F}_0$. *Moreover, if we choose an* $(\mathcal{O}/\mathfrak{p}^m)$*-basis* $\langle e_1, \dots, e_n \rangle$ *of* $(\mathfrak{p}^{-m}/\mathcal{O})^n$ *and define* $X_i := \alpha^{\mathrm{univ}}(e_i)$ $(1 \leq i \leq n)$ *where* α^{univ} *is the universal Drinfeld level* $\mathfrak{p}^m$*-structure on* $\widetilde{\Sigma} \otimes R_m$, *then* $X_1, \dots, X_n$ *is a set of regular parameters of* R_m.

As the functor $\mathscr{F}_m$ has a natural right action of $G_m := \mathrm{Aut}_{\mathcal{O}}\big((\mathfrak{p}^{-m}/\mathcal{O})^n\big) \cong GL_n(\mathcal{O}/\mathfrak{p}^m)$ by $[g] : (\Sigma, i, \alpha) \mapsto (\Sigma, i, \alpha \circ g)$ for $\forall g \in G_m$, the R_m has the corresponding left G_m-action as an R_0-algebra. As $[a] : (\Sigma, i, \alpha) \xrightarrow{\cong} (\Sigma, i \circ [a], \alpha \circ a)$ for $\forall a \in \mathcal{O}^\times$, the action of $\mathcal{D}^\times \times G_m$ on R_m factors through $\big(\mathcal{D}^\times \times G_m\big)/\mathcal{O}^\times$, where $\mathcal{O}^\times$ is embedded diagonally. As a Drinfeld level $\mathfrak{p}^{m+1}$-structure restricts to a Drinfeld level $\mathfrak{p}^m$-structure, we have a natural morphism of functors $\mathscr{F}_{m+1} \longrightarrow \mathscr{F}_m$, compatible with the action of G_m via the surjection $G_{m+1} \to G_m$. Therefore, the direct system $\{R_m\}$ acquires an action of $\big(\mathcal{D}^\times \times U\big)/\mathcal{O}^\times$, where $U := \varprojlim G_m = \mathrm{Aut}_{\mathcal{O}}\big((K/\mathcal{O})^n\big) \cong GL_n(\mathcal{O})$.

3.2.2. *Non-abelian Lubin-Tate theory.* **(a) Quasi-isogenies.** Let Σ be a formal $\mathcal{O}$-module over $(A, \mathfrak{m}) \in \mathscr{C}$ of height n. We write $\overline{\Sigma} := \Sigma \otimes_A \overline{A}$, and Σ_n for $\Sigma_n \otimes_{\overline{\mathbb{F}}_q} \overline{A}$, for simplicity. The set of all isomorphisms from Σ_n to $\overline{\Sigma}$:

$$\mathrm{Isom}(\Sigma_n, \overline{\Sigma}) := \{i : \Sigma_n \xrightarrow{\cong} \overline{\Sigma}\}$$

is a right $\mathcal{D}^\times$-torsor. Similarly, $\mathrm{Hom}(\Sigma_n, \overline{\Sigma})$ is a free right $\mathcal{D}$-module of rank 1, hence $\mathrm{Hom}(\Sigma_n, \overline{\Sigma}) \otimes_{\mathcal{O}} K$ is a free right D-module of rank 1, and

$$\mathrm{Qisog}(\Sigma_n, \overline{\Sigma}) := \mathrm{Hom}(\Sigma_n, \overline{\Sigma}) \otimes K - \{0\}$$

is a right $D^\times$-torsor containing $\mathrm{Isom}(\Sigma_n, \overline{\Sigma})$ as an $\mathcal{D}^\times$-subset. Any $i_0 \in \mathrm{Isom}(\Sigma_n, \overline{\Sigma})$ gives a bijection $D^\times \ni d \overset{\cong}{\longmapsto} i_0 \circ d \in \mathrm{Qisog}(\Sigma_n, \overline{\Sigma})$. The valuation $v : D^\times/\mathcal{D}^\times \longrightarrow \mathbb{Z}$ induces a surjection

$$\mathrm{ht} : \mathrm{Qisog}(\Sigma_n, \overline{\Sigma}) \ni i_0 \circ d \longmapsto v(d) \in \mathbb{Z},$$

which does not depend on the choice of i_0. For $j \in \mathbb{Z}$, an element $i \in \mathrm{Qisog}^{(j)}(\Sigma_n, \overline{\Sigma}) := \{i \in \mathrm{Qisog}(\Sigma_n, \overline{\Sigma}) \mid \mathrm{ht}(i) = j\}$ is called a *quasi-isogeny of height* j from Σ_n to $\overline{\Sigma}$. We have $\mathrm{Qisog}^{(0)}(\Sigma_n, \overline{\Sigma}) = \mathrm{Isom}(\Sigma_n, \overline{\Sigma})$, and any $d \in D^\times$ with $v(d) = j$ induces a bijection:

$$[d] : \mathrm{Isom}(\Sigma_n, \overline{\Sigma}) \ni i \overset{\cong}{\longmapsto} i \circ d \in \mathrm{Qisog}^{(j)}(\Sigma_n, \overline{\Sigma}).$$

(b) Action of $D^\times \times ZU$. Now for $m \geq 1$ and $j \in \mathbb{Z}$, we consider the functor $\mathscr{F}_m^{(j)} : \mathscr{C} \to (\mathrm{Sets})$ by sending $(A, \mathfrak{m})$ to the set of isomorphism classes of (Σ, i, α), where (Σ, i, α) is as in $\mathscr{F}_m$ but now $i \in \mathrm{Qisog}^{(j)}(\Sigma_n, \overline{\Sigma})$, instead of $i \in \mathrm{Isom}(\Sigma_n, \overline{\Sigma})$. We see that each $d \in D^\times$ with $v(d) = j$ gives an isomorphism of functors $[d] : \mathscr{F}_m = \mathscr{F}_m^{(0)} \cong \mathscr{F}_m^{(j)}$ via above bijection. Let $X_m^{(j)} := \mathrm{Spec}\, R_m^{(j)}$, where $R_m^{(j)} \in \mathscr{C}$ represents $\mathscr{F}_m^{(j)}$, and $X_m := \coprod_{j \in \mathbb{Z}} X_m^{(j)}$. Then the right $\mathcal{D}^\times$-action on $X_m^{(0)}$ naturally extends to the right $D^\times$-action on X_m: we have $[d] : (\Sigma, i, \alpha) \mapsto (\Sigma, i \circ d, \alpha)$ for $\forall d \in D^\times$. Hence the right action of $\big(\mathcal{D}^\times \times U\big)/\mathcal{O}^\times$ on $X_m^{(0)}$ extends to the right action of $\big(D^\times \times U\big)/\mathcal{O}^\times \cong \big(D^\times \times ZU\big)/K^\times$ on X_m, where $ZU := K^\times U \subset G := GL_n(K)$. In other words, $g \in Z = K^\times$ acts as $g^{-1} \in D^\times$.

(c) Weil descent. Let $\varphi := \mathrm{Frob}_K^{-1} : \widehat{\mathcal{O}} \xrightarrow{\cong} \widehat{\mathcal{O}}$ be the arithmetic Frobenius map, the unique lift of $\varphi : \overline{\mathbb{F}}_q \ni x \longmapsto x^q \in \overline{\mathbb{F}}_q$. For an $A \in \mathscr{C}$ with the structure morphism $\iota : \widehat{\mathcal{O}} \to A$, let:

(i) $A_\varphi \in \mathscr{C}$ be the ring A with the structure morphism $\iota \circ \varphi : \widehat{\mathcal{O}} \to A$.
(ii) ${}^\varphi\! A \in \mathscr{C}$ be the base change of A by φ, i.e. ${}^\varphi\! A := A \otimes_{\widehat{\mathcal{O}}, \varphi} \widehat{\mathcal{O}}$.

Then we have a canonical bijection $\mathrm{Hom}_{\mathscr{C}}({}^{\varphi}R, A) \cong \mathrm{Hom}_{\mathscr{C}}(R, A_{\varphi})$. In other words, if a functor $\mathscr{F} : \mathscr{C} \to (\text{Sets})$ is represented by R, then $\mathscr{F}^{\varphi} : \mathscr{C} \ni A \longmapsto \mathscr{F}(A_{\varphi}) \in (\text{Sets})$ is represented by ${}^{\varphi}R$. If $X = \operatorname{Spec} R$, we write $X^{\varphi} := \operatorname{Spec} {}^{\varphi}R$.

Let $F : \Sigma_n \to \Sigma_n^{\varphi}$ be the geometric Frobenius morphism X^q on Σ_n, where $\Sigma_n^{\varphi} := \Sigma_n \otimes_{\overline{\mathbb{F}}_q, \varphi} \overline{\mathbb{F}}_q$ is the base change of Σ_n by the arithmetic Frobenius φ; this F is a quasi-isogeny of height 1.

Now we consider the functor $\big(\mathscr{F}_m^{(j)}\big)^{\varphi}$. The set $\big(\mathscr{F}_m^{(j)}\big)^{\varphi}(A)$ for $A \in \mathscr{C}$ is the set of isomorphism classes of triples (Σ, i, α) defined over A_{φ}. First observe that, as the notion of formal $\mathcal{O}$-module over A refers only to the $\mathcal{O}$-algebra structure of A and as φ fixes $\mathcal{O}$, formal $\mathcal{O}$-modules over A and those over A_{φ} are the same thing. Similarly, the notion of Drinfeld level structure is insensitive to the change of base from A to A_{φ}. As for the quasi isogeny i of height j from $\Sigma_n \otimes_{\overline{\mathbb{F}}_q} \overline{A_{\varphi}}$ to $\Sigma \otimes_{A_{\varphi}} \overline{A_{\varphi}}$, we have $\Sigma_n \otimes_{\overline{\mathbb{F}}_q} \overline{A_{\varphi}} = \Sigma_n^{\varphi} \otimes_{\overline{\mathbb{F}}_q} \overline{A}$:

$$i \in \mathrm{Qisog}^{(j)}\big(\Sigma_n \otimes_{\overline{\mathbb{F}}_q} \overline{A_{\varphi}},\ \Sigma \otimes_{A_{\varphi}} \overline{A_{\varphi}}\big) = \mathrm{Qisog}^{(j)}\big(\Sigma_n^{\varphi} \otimes_{\overline{\mathbb{F}}_q} \overline{A},\ \Sigma \otimes_{A} \overline{A}\big).$$

Hence we can think of the triple (Σ, i, α) as defined over A, except that now $i \in \mathrm{Qisog}^{(j)}(\Sigma_n^{\varphi}, \overline{\Sigma})$. We can compose F with this i to get $i \circ F \in \mathrm{Qisog}^{(j+1)}(\Sigma_n, \overline{\Sigma})$. As before, composing a quasi-isogeny is a bijection:

$$\mathrm{Qisog}^{(j)}(\Sigma_n^{\varphi}, \overline{\Sigma}) \ni i \longmapsto i \circ F \in \mathrm{Qisog}^{(j+1)}(\Sigma_n, \overline{\Sigma}).$$

This defines an isomorphism of functors $[F] : \big(\mathscr{F}_m^{(j)}\big)^{\varphi} \longrightarrow \mathscr{F}_m^{(j+1)}$ for each $j \in \mathbb{Z}$, hence an isomorphism of schemes $[F] : \big(X_m^{(j)}\big)^{\varphi} \xrightarrow{\cong} X_m^{(j+1)}$, and finally $[F] : X_m^{\varphi} \xrightarrow{\cong} X_m$.

(d) Action of $D^{\times} \times G$. Now we vary $m \geq 0$ and consider the inverse system $X := \{X_m\}_{m \geq 0}$. We will extend the right action of $\big(D^{\times} \times ZU\big)/K^{\times}$ on X to a right action of $\big(D^{\times} \times G\big)/K^{\times}$. For this we need to consider formal $\mathcal{O}$-modules as Barsotti-Tate mO-modules.

It is enough to define the action $[g] : X \xrightarrow{\cong} X$ for $g \in G = GL_n(K)$ such that $g^{-1} \in M_n(\mathcal{O}) = \mathrm{End}_{\mathcal{O}}(K/\mathcal{O})$, because such g and Z generate G. We describe the image under $[g]$ of (Σ, i, α), defined over $(A, \mathfrak{m}) \in \mathscr{C}$. Consider the kernel $\operatorname{Ker} g^{-1}$ of $g^{-1} : K/\mathcal{O} \to K/\mathcal{O}$, which is a finite $\mathcal{O}$-submodule of $K/\mathcal{O}$. Let $\alpha(\operatorname{Ker} g^{-1})$ be the unique finite flat $\mathcal{O}$-submodule scheme of Σ which has $\alpha(\operatorname{Ker} g^{-1})$ as full set of sections. Consider the quotient $\Sigma' := \Sigma/\alpha(\operatorname{Ker} g^{-1})$, and let $\mathrm{can} : \Sigma \to \Sigma'$ be the canonical isogeny. Then there is a unique Drinfeld level structure α' on Σ' satisfying $\alpha' \circ g^{-1} = \mathrm{can} \circ \alpha$. Letting $\overline{\mathrm{can}} := \mathrm{can} \otimes_A \overline{A} : \Sigma_n \to \overline{\Sigma}$, we define:

$$[g] : (\Sigma, i, \alpha) \longmapsto (\Sigma',\ \overline{\mathrm{can}} \circ i,\ \alpha').$$

Note that $g \in U$ satisfies $g^{-1} \in M_n(\mathcal{O})$, and in this case $\operatorname{Ker} g^{-1} = 0,\ \Sigma' = \Sigma,\ \mathrm{can} = \mathrm{id}$ and $\alpha' = \alpha \circ g$ which is the action we defined previously. As for the elements $g \in Z = K^{\times}$ with $g^{-1} \in M_n(\mathcal{O})$, we have $g^{-1} \in \mathcal{O}$ and $\alpha(\operatorname{Ker} g^{-1}) = \Sigma[g^{-1}]$. There is an isomorphism $\psi : \Sigma \to \Sigma'$ such that $\psi \circ [g^{-1}] = \mathrm{can}$, and we have $(\psi \circ \alpha) \circ g^{-1} = \psi \circ [g^{-1}] \circ \alpha = \mathrm{can} \circ \alpha$, which shows $\alpha' = \psi \circ \alpha$. Therefore ψ gives an isomorphism

$(\Sigma, [g^{-1}]\circ i, \alpha) \longmapsto (\Sigma',\ \overline{\mathrm{can}}\circ i,\ \alpha')$, and as $[g^{-1}]\circ i = i \circ [g^{-1}]$, this action coincides with the previously defined action.

(e) Vanishing cycles. Now we consider the vanishing cycles. As each $X_m^{(0)}$ is known to be a complete local ring of an $\widehat{\mathcal{O}}$-scheme of finite type (see Section 4 or [St]), the vector space of ℓ-adic vanishing cycles at the closed point $x^{(0)} \in X_m^{(0)}$ for $\ell \neq p$:

$$\big(R^i\Psi\overline{\mathbb{Q}}_\ell\big)_{x^{(0)}} = H^i_{\mathrm{et}}\Big(X_m^{(0)} \times_{\widehat{\mathcal{O}}} \widehat{\overline{K}},\ \overline{\mathbb{Q}}_\ell\Big) \quad (0 \le i \le n-1)$$

is finite dimensional and has a left action by $I_K \times \big(\mathcal{D}^\times \times U\big)/\mathcal{O}^\times$. If we consider the vanishing cycles at the closed points $x^{(j)} \in X_m^{(j)}$ for each $j \in \mathbb{Z}$ and consider:

$$\Psi^i := \varinjlim_m \prod_{j\in\mathbb{Z}} \big(R^i\Psi\overline{\mathbb{Q}}_\ell\big)_{x^{(j)}} = \varinjlim_m H^i_{\mathrm{et}}\Big(X_m \times_{\widehat{\mathcal{O}}} \widehat{\overline{K}},\ \overline{\mathbb{Q}}_\ell\Big) \quad (0 \le i \le n-1)$$

then it has a left action by $I_K \times \big(D^\times \times G\big)/K^\times$.

4. Shimura varieties and local-global compatibility

4.1. **Global Langlands correspondence.** Let L be a number field (finite over $\mathbb{Q}$), n a positive integer, ℓ a fixed prime and $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ a fixed field isomorphism. We denote the absolute Galois group of L by $G_L = \mathrm{Gal}(\overline{L}/L)$. Conjectural global Langlands correspondence predicts a correspondence between (A) algebraic automorphic representation Π of $GL_n(\mathbb{A}_L)$ and (B) n-dimensional ℓ-adic Galois representation $R : G_L \to GL_n(\overline{\mathbb{Q}}_\ell)$ which is (1) almost everywhere unramified and (2) de Rham at ℓ. The cuspidals in (A) should correspond to irreducibles in (B). We are interested in cuspidal Π, and we denote the (conjectural) Galois representation attached to Π by $R = R_{\ell,\iota}(\Pi)$. Up to semisimplification, it is characterized by the property that the eigenvalues of Frobenius Frob_v equal the Satake parameters of Π_v for almost all places of v. One of the most general results in the direction $\Pi \longmapsto R$ is the one obtained by Kottwitz [K], Clozel [**?**] and Harris-Taylor [HT], which constructs the semisimple representation $R_{\ell,\iota}(\Pi)$ when L is an imaginary CM-field and Π is cuspidal, satisfying (1) Π is conjugate self-dual, (2) Π is regular algebraic and (3) there is a finite place x of L where Π_x is (essentially) square-integrable.

Theorem 4.1. ([HT], [TY]) *Let L be a CM-field, ℓ a prime and fix $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Let Π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ satisfying the three conditions above, and $R_{\ell,\iota}(\Pi)$ be the associated ℓ-adic representation of G_L. Then, for all finite place v of L not dividing ℓ:*

$$\iota\mathrm{WD}(R_{\ell,\iota}(\Pi)|_{G_{L_v}})^{F\text{-}ss} \cong \mathrm{rec}\big(\Pi_v^\vee \cdot |\det|_K^{\frac{1-n}{2}}\big)$$

as Weil-Deligne representations over $\mathbb{C}$ of W_{L_v}.

4.2. **Unitary Shimura varieties.** Now we introduce a class of Shimura varieties containg those studied in [HT]. Let F be a CM field, with complex conjugation c, of the form $F = EF^+$, where $F^+ \subset F$ is the fixed field of c and $E/\mathbb{Q}$ is imaginary quadratic. Let B be a simple algebra with center F and $\dim_F B = n^2$, with a positive involution $*$ with $*|_F = c$ and an alternating form $\langle , \rangle : B \times B \to \mathbb{Q}$ such that $\langle bx, y\rangle = \langle x, b^* y\rangle$ for $\forall b \in B$. Let G be the $\mathbb{Q}$-similitude group of $(B, \langle , \rangle)$. Then $G_0 := \operatorname{Ker}(G \to \mathbb{Q}^\times)$ is the restriction of scalars from a unitary group over F^+. We choose $\langle , \rangle$ so that $G_0(\mathbb{R}) \cong U(1, n-1) \times U(0, n)^{d-1}$, where $d := [F^+ : \mathbb{Q}]$. For each open compact subgroup $U \subset G(\mathbb{A}^\infty)$ small enough (where $\mathbb{A}^\infty := \widehat{\mathbb{Z}} \otimes \mathbb{Q}$), we define the Shimura variety X_U/F as a moduli of isogeny classes of quadruples $(A, \lambda, i, \eta U)$ of an abelian variety A of dimension dn^2, a polarization $\lambda : A \to A^\vee$, a ring homomorphism $i : B \to \operatorname{End}(A) \otimes \mathbb{Q}$, satisfying the Kottwitz conditon on Lie$A$ corresponding to G (see [**?**]) and $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$ for $\forall b \in B$, and a right U-orbit ηU of $B \otimes \mathbb{A}^\infty$-isomorphisms $\eta : B \otimes \mathbb{A}^\infty \longrightarrow VA := \Big(\varprojlim_N A[N]\Big) \otimes \mathbb{Q}$ which sends $\langle , \rangle$ to the λ-Weil pairing. Then X_U/F is a quasi-projective smooth variety of dimension $n-1$, which is projective if $d > 1$ or if B is a division algebra. We choose a place v of F lying over a prime p which splits in E. Then $G(\mathbb{Q}_p)$ is a product of $GL_n(F_v)$ and other factors, so set $G(\mathbb{A}^\infty) = G(\mathbb{A}^{\infty,v}) \times GL_n(F_v)$. We set $U = U^v \times \mathrm{Iw}_n$ where $U^v \subset G(\mathbb{A}^{\infty,v})$ and Iw_n is the open compact subgroup of $GL_n(\mathcal{O}_v)$ consisting of matrices which reduce to upper triangluar matrices modulo v. For $U_0 := U^v \times GL_n(\mathcal{O}_v)$ the X_{U_0} extends to a smooth scheme over $\mathcal{O}_v$ with a universal abelian scheme $\mathscr{A}/X_{U_0}$. Then $\mathcal{G} := \operatorname{diag}(1, 0, \ldots, 0)\mathscr{A}[v^\infty]$ is a 1-dimensional Barsotti-Tate $\mathcal{O}_v$-module of $\mathcal{O}_v$-height n, i.e. $\mathcal{G}[v]$ is a finite flat group scheme of degree $|k(v)|^n$. The moduli of chain of n isogenies each of degree $|k(v)|$ factoring $\mathcal{G} \to \mathcal{G}/\mathcal{G}[v]$ gives a regular strictly semistable model of X_U over $\mathcal{O}_v$, which is finite flat over X_{U_0}.

4.3. **Weight spectral sequence.** Let K be a complete discrete valuation field with a finite residue field k and the ring of integers $\mathcal{O}_K$. Let X be a proper strictly semistable scheme of relative dimension $n-1$ over $\mathcal{O}_K$. Then its special fiber $Y := X \times_{\mathcal{O}_K} k$ is written as $Y = \bigcup_{i \in \Delta} Y_i$ with $\Delta := \{1, ..., t\}$ and Y_i proper smooth over k, where Y_i and Y_j intersect transversally for $i \neq j$. Let $Y_I := \bigcap_{i \in I} Y_i$ for $I \subset \Delta$, which is proper smooth over k of dimension $n - |I|$ if not empty, and $Y^{(m)} := \coprod_{|I|=m} Y_I$ for $1 \leq m \leq n$. For a prime $\ell \neq \operatorname{char} k$, the weight spectral sequence reads

$$E_1^{i,j} := \bigoplus_{s \geq \max(0,-i)} H^{j-2s}\big(Y^{(i+2s+1)} \times_k \overline{k},\ \overline{\mathbb{Q}}_\ell(-s)\big) \Longrightarrow H^{i+j}(X \times_K \overline{K},\ \overline{\mathbb{Q}}_\ell).$$

4.3.1. *Degeneration of weight spectral sequence.* Now the special fiber $Y := X_U \otimes_{\mathcal{O}_v} k(v)$ is written as $Y = \bigcup_{1 \leq i \leq n} Y_i$, where Y_i, smooth over $k(v)$, is the locus where the i-th isogeny in the chain induces zero map on the Lie algebra. This moduli interpretation allows us to apply Theorem 4.2 to X_U, when it is proper, and the Hecke correspondences generating the local Hecke algebra $\mathcal{H}_n := \overline{\mathbb{Q}}_\ell\big[\mathrm{Iw}_n \backslash GL_n(F_v)/\mathrm{Iw}_n\big]$. It is generated by the generators $w_1, \ldots, w_{n-1}$ and $T_1^\pm, \ldots, T_n^\pm$ of the extended affine Weyl group, subject to certain relations (Bernstein-Zelevinsky presentation). It naturally contains the Iwahori

Hecke algebra of Levi subgroups, say $\mathcal{H}_m \otimes \mathcal{H}_{n-m}$ of $GL_m \times GL_{n-m}$, which is generated by the above set of generators minus w_m, and makes $\mathcal{H}_n$ into a finite $\mathcal{H}_m \otimes \mathcal{H}_{n-m}$-algebra of dimension $\binom{n}{m}$. Now we refine the computation done in [TY]: we compute $H^*(Y^{(m)})$ as $\mathcal{H}_n$-module. By dividing it into open strata, we see that $H^*(Y^{(m)})$ (the alternating sum in the Grothendieck group) is the sum of $\mathcal{H}_n \otimes H^*_c(Y^0_{I_s})$ for $m \leq s \leq n$, where $I_s := \{1, \ldots, s\}$ and $Y^0_{I_s} := Y_{I_s} - \bigcup_{I_s \subset I \neq I_s} Y_I$ is the open stratum of Y_{I_s}, and the tensor product is over $\mathcal{H}_m \otimes \mathcal{H}_{s-m} \otimes \mathcal{H}_{n-s}$. Now, the action of $\mathcal{H}_m \otimes \mathcal{H}_{s-m} \otimes \mathcal{H}_{n-s}$ on $H^*_c(Y^0_{I_s})$ is given as follows: the action of $\mathcal{H}_m \otimes \mathcal{H}_{s-m}$ is computed locally by Theorem 4.2, and the action of $\mathcal{H}_{n-s}$ is computed by counting of points on Igusa varieties via trace formula ([**?**], this is where we need global assumptions). The action of $\mathcal{H}_m \otimes \mathcal{H}_{s-m}$ is roughly given by $\mathrm{St}_m \otimes \mathrm{Tr}_{s-m}$, with some unramified twists corresponding to the Frobenius action, where St_n is a 1-dimensional $\mathcal{H}_n$-module given by $w_i \mapsto -1$ and $T_i \mapsto 1$, similarly Tr_n is a 1-dimensional $\mathcal{H}_n$-module given by $w_i \mapsto q := |k(v)|$ and $T_i \mapsto q^{i(n-i)}$, and $\otimes$ denotes the product corresponding to non-normalized induction. When we look at the E_1-term $H^*(Y^{(m)})$ of the weight spectral sequence after taking the limit with respect to U^v, making $H^*(Y^{(m)})$ into a $G(\mathbb{A}^{\infty,v}) \times \mathcal{H}_n \times \mathrm{Frob}^{\mathbb{Z}}$-module, its $\pi^{\infty,v}$-isotypic component, where $\pi = \pi^{\infty,v} \times \pi_v$ is a cuspidal automorphic respresentation of $G(\mathbb{A}^\infty)$, recovers $\pi_v^{\mathrm{Iw}_v}$ as $\mathcal{H}_n$-module and is pure of weight $n-m$ as $\mathrm{Frob}^{\mathbb{Z}}$-module. Two things are used: (1) the global result on the cohomology of $Y^0_{I_s}$ shows that as $\mathcal{H}_{n-s}$-module $H^*(Y^0_{I_s})[\pi^{\infty,v}]$ is essentially the (Iwahori invariants of) the Jacquet module of π_v to $GL_s \times GL_{n-s}$, and (2) the cancellation $\sum_{m=0}^{s}(-1)^m \mathrm{St}_m \otimes \mathrm{Tr}_{s-m} = 0$ in the Grothendieck group of $\mathcal{H}_s$-modules.

4.3.2. *Weight spectral sequence and Hecke correspondences.* Now let Γ be an algebraic correspondence on X (namely an n-dimensional cycle on $X \times_{\mathcal{O}_K} X$) such that two projection maps $\Gamma \to X$ are both finite. We are interested in the action $[\Gamma_K]^* := \mathrm{pr}_{1*} \circ ([\Gamma_K]\cup) \circ \mathrm{pr}_2^*$ of $\Gamma_K := \Gamma \times_{\mathcal{O}_K} K$ on $H^*(X \times_K \overline{K}, \overline{\mathbb{Q}}_\ell)$. Let $Y_{I,J} := Y_I \times_k Y_J$ for $I, J \subset \Delta$, and write $Y_{i,j} := Y_{\{i\},\{j\}}$. Let $(X \times_{\mathcal{O}_K} X)_{\mathrm{sm}}$ be the smooth locus of the morphism $X \times_{\mathcal{O}_K} X \longrightarrow \operatorname{Spec} \mathcal{O}_K$, and let $Y^0_{i,j} := Y_{i,j} \cap (X \times_{\mathcal{O}_K} X)_{\mathrm{sm}}$. Then $Y^0_{i,j}$ is a Cartier divisor of $(X \times_{\mathcal{O}_K} X)_{\mathrm{sm}}$.

Theorem 4.2. *There is a unique collection $\{\Gamma_{I,J}\}$ of cycles $\Gamma_{I,J}$ on $Y_{I,J}$ for all pairs (I,J) with $|I| = |J|$, satisfying the following two conditions.*

(i) *$\Gamma_{i,j}$ is the closure of the cycle $\Gamma^0_{i,j} := Y^0_{i,j} \cdot \Gamma|_{(X \times_{\mathcal{O}_K} X)_{\mathrm{sm}}}$ in $Y_{i,j}$.*

(ii) *When $|I| = |J| + 1 = m$ and $I = \{i_1, ..., i_m\}$, $J = \{j_1, \ldots, j_{m-1}\}$ are in increasing order, there is an equality:*

$$\sum_{h=1}^{m} (-1)^h \, Y_{I,J} \cdot \Gamma_{I \setminus \{i_h\}, J} = \sum_{j \in \Delta \setminus J} (-1)^{h(j)} \, \Gamma_{I, J \cup \{j\}}$$

of $(n-m)$-dimensional cycles on $Y_{I,J}$, where $1 \leq h(j) \leq m$ is determined by $j_{h(j)-1} < j < j_{h(j)}$ (set $j_m := \infty$).

Then setting $\Gamma^{(m)} := \coprod_{|I|=|J|=m} \Gamma_{I,J}$ *as an* $(n-m)$*-dimensional cycle on* $Y^{(m)} \times_k Y^{(m)}$ *for* $1 \leq m \leq n$*, the action* $\oplus[\Gamma^{(i+2s+1)}]^*$ *on* $E_1^{i,j}$ *is compatible with the action* $[\Gamma_K]^*$ *on* $H^{i+j}(X \times_K \overline{K}, \overline{\mathbb{Q}}_\ell)$.

For the proof of this theorem, we build on the construction of [Sa], except that we eliminate the semistable resolution of $X \times_{\mathcal{O}_K} X$ from the description of the cycles $\Gamma_{I,J}$, in order to apply the formula to the Shimura varieties where the cycles have concrete moduli interpretation. For this we also need the cycles, not only cycle classes.

4.4. **Results of Boyer and Dat.**

References

[B] P. Boyer, *Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura simples*, preprint, `http://www.institut.math.jussieu.fr/~boyer/`.

[Ca] H. Carayol, *Non-abelian Lubin-Tate theory*, in: *Automorphic Forms, Shimura Varieties, and L-functions* (Academic Press, 1990), Vol. II, pp.15-39.

[Cl] L. Clozel, *Représentations Galoisiennes associées aux représentations automorphes autoduals de GL(n)*, Publ. Math. IHES, **73** (1991), 97–145.

[Co] R. Coleman, *Division values in local fields*, Invent. Math. **53** (1979), 91-116.

[DL] P. Deligne, G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. **103** (1976), 103-161.

[Dr] V. Drinfeld, *Elliptic modules*, Math. USSR Sbornik **23**–4 (1974), 561-592.

[Fa] G. Faltings, *A relation between two moduli spaces studied by Drinfeld*, in: Âlgebraic Number Theory and Algebraic Geometry, Contemp. Math., vol. 300, AMS, Providence (2002), 115-129.

[FGL] L. Fargues, A. Genestier, V. Lafforgue, *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, to appear in Progr. Math., Birkhäuser (2007).

[GH] B. Gross, M. Hopkins, *Equivariant vector bundles on the Lubin-Tate moduli space*, in: *Topology and representation theory* (Evanston, IL, 1992), Contemp. Math., vol. 158, AMS, Providence (1994), 23–88.

[He1] G. Henniart, *Une preuve simple des conjectures de Langlands pour GL(n) sure un corps p-adiques*, Invent. Math. **139** (2000), 439–455.

[He2] G. Henniart, *Une caracterisation de la correspondance de Langlands locale pour GL(n)*, BullṠoc. Math. Fr. **130**-4 (2002), 587-602.

[HT] M. Harris, R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties*, Ann. of Math. Studies **151**, Princeton Univ. Press, Princeton-Oxford, 2001.

[Iw] K. Iwasawa, *Local Class Field Theory*, Oxford Univ. Press, 1986.

[K] R. Kottwitz, *On the λ-adic representations associtated to some simple Shimura varieties*, Invent. Math. **108** (1992), 653–665.

[LT] J. Lubin, J. Tate, *Formal complex multiplication in local fields*, Ann. Math. **81** (1965), 380-387.

[LT2] J. Lubin, J. Tate, *Formalmoduli for one-parameter formal Lie groups*, Bull. Soc. Math. Fr. **94** (1966), 49–59.

[RZ] M. Rapoport, T. Zink, *Über die lokale Zetafunktion von Shimuravarietäten, Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. **68** (1982), no. 1, 21–101.

[Sa] T. Saito, *Weight spectral sequences and independence of ℓ*, J. Inst. Math. Jussieu **2** (2003), 583–634.

[Se] J.-P. Serre, *Corps Locaux*, Hermann, Paris, 1962.

[St] M. Strauch, *Deformation spaces of one-dimensional formal modules and their cohomology*, preprint, 2006.

[Yo1] T. Yoshida, *On non-abelian Lubin-Tate theory via vanishing cycles*, to appear in Annales de l'Insitut Fourier. `http://arxiv.org/abs/math/0404375`

[Yo2] T. Yoshida, *Weight spectral sequence and Hecke correspondence on Shimura varieties*, thesis, to be submitted.

[Yo3] T. Yoshida, *Local class field theory via Lubin-Tate theory*, to appear in Annales de la Faculte des Sciences de Toulouse. `http://arxiv.org/abs/math/0606108`

[TY] R. Taylor, T. Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc., **20**-2 (2007), 467-493.

Harvard University, Department of Mathematics, 1 Oxford Street, Cambridge, MA 02138, USA