Math 264 — Homework #3

Due Wednesday, February 20, 2002

1. Find the expected number 2,875 of lines on a general quintic 3-fold $X \subset \mathbb{P}^4$.

2. The standard dimension count leads us to expect that a general quartic threefold $X \subset \mathbb{P}^4$ will contain a one-parameter family of lines. (In fact, there's a cute deformation theoretic argument that *every* smooth quartic does—try to prove this.) Assuming that the dimension of the family $F_1(X)$ of lines on X is exactly 1, find the degree of the surface $S \subset X \subset \mathbb{P}^4$ swept out by these lines.

3. The techniques introduced here can also be used to find the expected class of the variety $F_k(X)$ of k-planes on a hypersurface $X \subset \mathbb{P}^n$ of degree d for any k. As a first example, find the number of 2-planes on a quadric hypersurface $Q \subset \mathbb{P}^4$.

This problem has additional significance since the answer (as you'll see when you do it) is 0. This means that a quadric hypersurface in $\mathbb{P}^4$ either has no 2-planes on it, or a positive-dimensional family of them. Since we know that the incidence correspondence

$$\Phi = \{(X, \Lambda) : \Lambda \subset X\} \subset \mathbb{P}^{14} \times \mathbb{G}(2, 4)$$

(where $\mathbb{P}^{14}$ is the space of quadrics in $\mathbb{P}^4$) has dimension 14, the latter cannot be the case generically.

In fact, a smooth quadric doesn't contain any 2-planes, while a singular (but irreducible) one necessarily contains a one-parameter family of them (check this!). Thus the projection map $\Phi \to \mathbb{P}^N$ fails to be dimensionally transverse in this case. Can you find any other cases where this happens?

4. Let $\mathbb{P}^{19}$ be the projective space of cubic surfaces in $\mathbb{P}^3$, and let

$$\Phi = \{(S, \Lambda) : \Lambda \subset S\} \subset \mathbb{P}^{19} \times \mathbb{G}(1, 3).$$

Find the class of Φ, and use this information to answer the question: if

$$\{S_\lambda = V(\lambda_0 F + \lambda_1 G) \subset \mathbb{P}^3\}_{[\lambda_0, \lambda_1] \in \mathbb{P}^1}$$

is a general pencil of cubic surfaces in $\mathbb{P}^3$, what is the degree of the surface in $\mathbb{P}^3$ swept out by the lines on the surfaces S_λ?

5. In the same vein as the last problem, let $\mathbb{P}^{34}$ be the projective space of quartic surfaces in $\mathbb{P}^3$ and let $\Phi = \{(S, \Lambda) : \Lambda \subset S\} \subset \mathbb{P}^{34} \times \mathbb{G}(1, 3)$ be the analogous incidence correspondence. Find the class of Φ, and use this information to answer the question: if $\{S_\lambda\}_{\lambda \in \mathbb{P}^1}$ is a general pencil of quartic surfaces in $\mathbb{P}^3$, how many of the surfaces S_λ contain a line?