Math 264 Tentative syllabus

I. Enumerative Geometry

A. Basic definitions

a) Basic definitions: Chow groups, intersection products, Chow ring

b) comparison with other cycle theories. Our ignorance of Chow groups in general; our indifference to this fact

c) behavior under push-forward and pullback. other basic properties?

B. $G(1,3)$:

a) Schubert cycles form a basis (do it by the decomp into affine spaces?) (could just prove generation and leave indep for an application of intersections).

b) compute pairwise int in comp dim by choosing transversal reps; use tangent space to the Grassmannian and Schubert cycles (quote from Harris) here and later.

c) triple intersections, general products, again by transversal reps.

d) Applications: the 2 lines meeting 4 lines; lines meeting 4 curves

e) secant lines to a curve

f) number of common chords to two curves

g) mention: problem of trisecant lines (will do as application of correspondences; again as application of specialization/deformation techniques, then again as application of GRR and Porteous)

C. $G(k,n)$:

a) Definition of Schubert cycles in general

b) Compute pairwise int in comp dim. Observe: degree of a curve on Grassmannian = degree of union of corresponding planes in projective space

c) Do triple intersection involving the hyperplane class σ_1; deduce Pieri's formula for σ_1 (and state it in general); derive formula for degree of Grassmannian in Plücker embedding

d) Other formulas: state Giambelli, Littlewood-Richardson; give argument for either?

D. Characteristic classes and easy applications

a) Definition of chern classes and Segre classes as degeneracy loci. Proof of independence of choice of sections (how?); alternative definitions?

b) Basic properties of Chern classes: Chern classes of dual bundles; Whitney product; splitting principle and application to chern classes of tensor product with line bundle; Chern classes of universal bundles and tangent bundle of projective space;

c) Application: number of singular elements of a pencil of curves in $\mathbf{P}^2$; more generally number of singular elts of a pencil on any variety

d) Chern classes of symmetric products and application: number of lines on a cubic surface in $\mathbf{P}^3$ (more generally, number of lines on hypersurface of degree $2n-3$ in $\mathbf{P}^n$?) Number of quartic surfaces in a pencil possessing a line.

e) Lines having contact of given order with a surface: lines on a cubic surface redux

f) Projective bundles: description of Chow ring; the universal line bundle and the Segre classes. Application: the parameter space of conics in $\mathbf{P}^3$ and the number of conics meeting 8 lines. tangent bundle to projective bundle? (we can combine the last two to derive a Porteous-free proof of existence of g^1_d's)

g) other examples: null-correlation bundle on $\mathbf{P}^3$; Horrocks-Mumford?

E. <u>Parameter spaces I: Compactification</u>

a) Original (erroneous) calculation of conics tangent to five conics

b) Solution: correct calculation in the space of complete conics

c) Chow rings of blow-ups; another description of Chow ring of space of complete conics

F. <u>Parameter spaces II: Linearization</u>

a) Problem of finding curves with cusps (resp. tacnodes) in a two- (resp. three-) dimensional linear series on a surface; solution via passage to the projectivized tangent bundle to the surface

G. <u>Excess intersection formulas</u>

a) description of formula and interpretation as "points absorbed" in family of transverse intersections specializing to excess one. Use degeneracy of sections of the normal bundle corresponding to the approaching families.

b) Simple applications: examples in the plane; intersection of three surfaces containing a line; verification via intersection numbers on surfaces

c) the number of conics tangent to five conics via excess intersection formula

H. <u>Porteous' formula</u>

a) description of formula; immediate application to degrees of determinantal varieties

b) proof that there are no smooth hypersurface containing a scroll (of dimension $\geq n/2$ in $\mathbf{P}^n$)

c) secant planes to rational curves (mention general case requires either correspondences or GRR)

d) Chow ring of symmetric products of curves; proof of existence half of Brill-Noether

I. <u>Correspondences on curves (?)</u>

a) basic definitions and formulas: valence, united points, common points, double points

b) Plücker formulas for plane curves via correspondences

c) secant planes to curves of any genus: degree of surface of trisecant lines (left from second section); number of stationary trisecants and quadrisecant lines.

J. <u>Double point formula</u>: Self intersection as top chern class of the normal bundle; double point formula; application?

Part II. Specialization and deformation arguments

A. Specialization to convenient position
- a) Bézout's Theorem;
- b) Classical derivation of the Schubert calculus
- c) Poincaré relation via specialization to hyperelliptic curves

B. Specialization with degeneration
- a) Noether-Lefschetz via reducible surfaces;
- b) Nonexistence part of Brill-Noether via cuspidal curves and Plücker;
- c) Gröbner bases and connectedness of the Hilbert scheme following Pardue

C. First-order deformation of subschemes = $H^0(N)$ and applications
- a) Dimension of the Fano scheme of linear spaces on a general hypersurface. In the case of lines, the normal bundle is generically balanced
- b) Every smooth quartic in P^4 has the right dimensional family of lines (could prove that this curve of lines is smooth in the generic case; also on the Fermat quartic it is everywhere non-reduced!)
- c) dimension of the chow variety of curves
- d) when is the general 0-cycle in $\mathbf{P}^n$ a hyperplane section of a curve in $\mathbf{P}^{n+1}$?

D. First-order deformations in general
- a) $H^1(T_X)$ as tan space to defos among *analytic* varieties. Dimension of the moduli space of curves. A deformation of a hypersurface is a hypersurface except for plane curves of degree 5 or more, quartics in P^3.
- b) Variations: Deformations of morphisms fixing the domain

$$H^0(T \mid X) \to H^0(N_{Y/X}) \to H^1(T_Y)$$

deformations of variety plus line bundle; of variety plus divisor...????

E. Dimension-based deformation arguments
- a) ??? (simple example)
- b) Limit linear series. the regeneration theorem & existence of Weierstrass points. (Should this really be here?)

E. Higher order deformations and obstructions
- a) obstruction groups associated to various deformation problems (for example, $H^1(N)$, $H^2(T)$); obstruction maps as obstruction to extending finite-order deformations
- b) existence of versal deformation spaces, smooth of the right dimension when obstruction group is zero. Examples: curves.
- c) dimension of deformation space is $\geq$ the dimension of the space of first-order deformations minus the dimension of the space of obstructions

(Note: this is necessarily going to a "survey chapter," that is, a lot of statements with references to Vistoli (for example) for proofs)

F. Bend-and-break (?):

a) rationally connected varieties; proofs (if feasible) of equivalence of conditions for a variety to be rationally connected and of the theorem that Fano varieties are rationally connected

b) irreducibility of spaces of rational curves on hypersurfaces (following Jason Starr)

III. Topology of algebraic varieties and Hodge theory

A. Generalized Riemann-Hurwitz and applications

a) number of singular elements in a pencil of curves in $\mathbf{P}^2$; verification by direct calculation

b) number of singular elements in a pencil of divisors on a variety in general

c) number of cuspidal curves in a net in $\mathbf{P}^2$ (or on a surface in general?); comparison with earlier method

B. Lefschetz hyperplane theorem and applications

a) universal hypersurfaces don't have points

b) every subvariety of more than half the dimension of a smooth hypersurface is homologous to a complete intersection (aka no hypersurface containing a scroll)

C. Basic definitions and theorems of Hodge theory

b) the Hodge theorem, Kahler condition and the resulting Hodge decomposition

b) the Lefschetz decomposition and index thereom, with applications

D. Hodge structures of hypersurfaces

a) the residue map and the description of Hodge groups of smooth hypersurfaces as spaces of polynomials mod the Jacobian ideal

b) infinitesimal variations and applications (for example, surfaces on general cubic fourfolds)

c) the Noether-Lefschetz theorem again, with estimates on codimension

E. Intermediate Jacobians & irrationality of cubic threefolds

a) Intermediate Jacobian of a threefold and the Abel-Jacobi map; when the intermediate jacobian is algebraic; discussion of intermediate Jacobians of cubic threefolds in particular

b) Geometry of the scheme S of lines on a cubic threefold and of the abel-jacobi map (for example, the differential of the map)

c) the theta divisor of the intermediate Jacobian J of a cubic threefold is the image of $S \times S$; we deduce that J is not the Jacobian a curve and hence the cubic threefold is irrational

d) (time permitting): what the intermediate Jacobian of a cubic threefold is (the Prym of a double cover of a plane quintic); conic bundles in general

F. Further topics

Monodromy (the Picard-Lefschetz transformation); Mixed Hodge structures; what is a motive? (can we find applications?)

IV. Cohomology

A. Grothendieck-Riemann-Roch and applications

a) Chern classes of vector bundles on symmetric powers of curves & application to formulas for secant planes; there must be more elementary applications)

B. Cohomology and base change

a) statement and examples of the basic theorem (again, there must be elementary applications)