# Abstract for the dissertation "Hecke eigenforms and the arithmetic of singular K3 surfaces"

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The dissertation investigates the arithmetic of singular K3 surfaces. It also gives an extensive analysis of Hecke eigenforms with rational coefficients and complex multiplication.

The thesis starts with a brief introduction on K3 surfaces. I emphasize the notion of singular K3 surfaces over $\mathbb{C}$, i.e. K3 surfaces with maximal Picard number $\varrho = 20$. In particular, I recall the theory of Shioda-Inose [S-I] and Livné's theorem that a singular K3 surface over $\mathbb{Q}$ is modular [L]. I then formulate the question which newforms might actually be associated to singular K3 surfaces over $\mathbb{Q}$.

This motivates the study of Hecke eigenforms with rational Fourier coefficients and complex multiplication (CM) of arbitrary weight. This investigation is the content of the second chapter of the thesis. The main result is the following

**Theorem**
*For fixed weight $k$, the newforms with rational Fourier coefficients and CM are, up to twisting, in 1:1-correspondence with the imaginary quadratic fields $K$ with exponent dividing $k-1$. This can be achieved by mapping a newform to its CM-field.*

The theorem is established using the Größencharakter associated to a newform with CM by Ribet [R]. Due to work of Stark [St] and Weinberger [We], it implies

**Corollary**
*Up to twisting, there are only finitely many CM-newforms with rational coefficients and weight $2, 3,$ or $4$. For arbitrary fixed weight, this holds subject to the Generalized Riemann Hypothesis.*

The corollary is supplemented by explicit tables for weights 3 and 4.

The third chapter is devoted to finding singular K3 surfaces over $\mathbb{Q}$. These are derived from rational elliptic surfaces by base change or other manipulation of the Weierstrass equation. All resulting fibrations are extremal. Together with other known examples, this approach enables me to find corresponding elliptic K3 surfaces over $\mathbb{Q}$ for 24 out of the 65 weight 3 forms from the list.

In the final chapter, I discuss one particular singular K3 surface in detail. This admits an extremal elliptic fibration with [1,1,1,12,3*] configuration. I determine the associated newform of level 27 and show that the Néron-Severi group $NS(X)$ can be generated by divisors over $\mathbb{Q}$. This gives a counterexample to a claim of Shioda [Sh2, Thm. 1]. One special property of this surface is the good reduction at 2. This reduction is studied in detail as is the corresponding surface over $\mathbb{F}_3$. I verify conjectures of Tate, Shioda and Artin for these cases.

Finally, I comment on the twists of the surface. These produce all weight 3 newforms with rational coefficients and CM by $\mathbb{Q}(\sqrt{-3})$. An analogous result can be derived for the CM-field $\mathbb{Q}(\sqrt{-1})$, for instance using the Fermat quartic in $\mathbb{P}^3$. Since a general Weierstrass equation trivially admits quadratic twisting, I deduce the

**Proposition**
*Let $f$ be one of the 24 newforms of weight 3 with rational coefficients for which we know an associated elliptic singular K3 surface over $\mathbb{Q}$. Then any twist of $f$ can be realized geometrically as an extremal elliptic K3 fibration over $\mathbb{Q}$.*

# References


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