Unit 38: Geometries and Fields

Lecture

38.1. Integral theorems deal with geometries $G$ and fields $F$. Integration pairs them up and gives the Stokes theorem

\[ \int_G dF = \int_{\partial G} F \]

It involves the boundary $\partial G$ of $G$ and the exterior derivative $dF$ of $F$. One can classify the theorems by looking at the dimension $n$ of space and the dimension $m$ of the object we are integrating over. In dimension $n$, there are $n$ theorems:

38.2. The Fundamental theorem of line integrals is a theorem about the gradient $\nabla f$. It tells that if $C$ is a curve going from $A$ to $B$ and $f$ is a function (that is a 0-form), then

**Theorem:** \[ \int_C \nabla f \cdot dr = f(B) - f(A) \]

In calculus we write the 1-form as a column vector field $\nabla f$. It actually is a 1-form $F = df$, a field which attaches a row vector to every point. If the 1-form is evaluated at $r'(t)$ one gets $df(r(t))(r'(t))$ which is the matrix product. We integrate then the pull back of the 1-form on the interval $[a, b]$. It is the switch from row vectors to column vectors which leads to the dot product $\nabla f(r(t)) \cdot r'(t)$. For closed curves, the line integral is zero. It follows also that integration is path independent.

38.3. Green’s theorem tells that if $G \subset \mathbb{R}^2$ is a region bound by a curve $C$ having $G$ to the left, then

**Theorem:** \[ \iint_G \text{curl}(F) \, dxdy = \oint_C F \cdot dr \]
In the language of forms, \( F = Pdx + Qdy \) is a 1-form and \( dF = (P_x dx + P_y dy)dx + (Q_x dx + Q_y dy)dy = (Q_x - P_y)dx dy \) is a 2-form. We write this 2-form as \( Q_x - P_y \) and treat it as a scalar function even so this is not the same as a 0-form, which is a scalar function. If \( \text{curl}(F) = 0 \) everywhere in \( \mathbb{R}^2 \) then \( F \) is a gradient field.

38.4. **Stokes theorem** tells that if \( S \) is a surface with boundary \( C \) oriented to have \( S \) to the left and \( F \) is a vector field, then

\[
\int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr
\]

In the general frame work, the field \( F = Pdx + Qdy + Rdz \) is a 1-form and the 2-form \( dF = (P_x dx + P_y dy + P_z dz)dx + (Q_x dx + Q_y dy + Q_z dz)dy + (R_x dx + R_y dy + R_z dz)dz = (Q_x - P_y)dx dy + (R_y - Q_z)dy dz + (P_z - R_x)dz dx \) is written as a column vector field \( \text{curl}(F) = [R_y - Q_z, P_z - R_x, Q_x - P_y]^T \). To understand the flux integral, we need to see what a bilinear form like \( dx dy \) does on the pair of vectors \( r_u, r_v \). In the case \( dx dy \) we have \( dx dy (r_u, r_v) = x_u y_v - y_u x_v \) which is the third component of the cross product \( r_u \times r_v \) with \( r_u = [x_u, y_u, z_u]^T \). Integrating \( dF \) over \( S \) is the same as integrating the dot product of \( \text{curl}(F) \cdot r_u \times r_v \). Stokes theorem implies that the flux of the curl of \( F \) only depends on the boundary of \( S \). In particular, the flux of the curl through a closed surface is zero because the boundary is empty.
38.5. **Gauss theorem**: if the surface $S$ bounds a solid $E$ in space, is oriented outwards, and $F$ is a vector field, then

\[ \iiint_E \text{div}(F) \, dV = \iint_S F \cdot dS \]

Gauss theorem deals with a 2-form $F = P \, dydz + Q \, dzdx + R \, dxdy$, but because a 2-form has three components, we can write it as a **vector field** $F = [P, Q, R]^T$. We have computed $dF = (P_x \, dx + P_y \, dy + Q_z \, dz) dydz + (Q_x \, dx + Q_y \, dy + Q_z \, dz) dzdx + (R_x \, dx + R_y \, dy + R_z \, dz) dxdy$, where only the terms $P_z dx dy dz + Q_y dy dz dx + R_x dxdy = (P_x + Q_y + R_z) dxdydz$ survive which we associate again with the scalar function $\text{div}(F) = P_x + Q_y + R_z$. The integral of a 3-form over a 3-solid is the usual triple integral. For a divergence free vector field $F$, the flux through a closed surface is zero. Divergence-free fields are also called **incompressible** or **source free**.

**Remarks**

38.6. We see why the 3 dimensional case looks confusing at first. We have three theorems which look very different. This type of confusion is common in science: we put things in the same bucket which actually are different: it is only in 3 dimensions that 1-forms and 2-forms can be identified. Actually, more is mixed up: not only are 1-forms and 2-forms identified, they are also written as vector fields which are $T^1_0$ tensor fields. From the tensor calculus point of view, we identify the three spaces $T^1_0(E) = E, T^0_1(E) = \Lambda^1(E) = E^\ast$ and $\Lambda^2(E) \subset T^2_0$. While we can still always identify vector fields with 1-forms, this identification in a general non-flat space will depend on the metric. In $\mathbb{R}^4$, the 2-forms have dimension 6 and can no more be written as a vector. One still does. The electro-magnetic $F$ is a 2-form in $\mathbb{R}^4$ which we write as a pair of two time-dependent vector fields, the electric field $E$ and the magnetic field $B$.

38.7. Geometries and fields are remarkably similar. On geometries, the **boundary operation** $\delta$ satisfies $\delta \circ \delta = 0$. On fields the **derivative operation** $d$ satisfies $d \circ d = 0$. ‘Geometries” as well as “fields” come with an **orientation**: $r_u \times r_v = - r_v \times r_u, dxdy = - dydx$. The operations $d$ and $\delta$ look different because calculus deals with smooth things like curves or surfaces leading to generalized functions. In **quantum calculus** they are thickened up and $d, \delta$ defined without limit. Fields and geometries then become indistinguishable elements in a Hilbert space. The exterior derivative $d$ has as an adjoint $\delta = d^\ast$ which is the boundary operator. It is a kind of quantum field theory as $d$ generates while $d^\ast$ destroys a “particle”. $d^2 = \delta^2 = 0$ is a “Pauli exclusion”.

38.8. We can spin this further: a **$m$-manifold** $S$ is the image of a parametrization $r : G \subset \mathbb{R}^m \to \mathbb{R}^n$. The Jacobian $dr$ is a **dual $m$-form**, the exterior product of the $m$ vectors $dr_{u_1}$ up to $dr_{u_m}$ (think of $m$ column vectors attached to $r(u) \in S$). If we take a map $s : S \subset \mathbb{R}^n \to \mathbb{R}^m$ and look at $F = ds$, we can think of it as a $m$-form $F$ (think of $m$ row vectors attached to each point $x$ in $\mathbb{R}^n$). The map $s$ defines $m \times n$ Jacobian $ds(x)$, while the Jacobian $dr(u)$ is the $n \times m$ matrix. Cauchy-Binet shows that the flux of $F$ through $r(G) = S$ is the integral $\int_G F = \int_G \det(ds(r(u))) dr(u) \, du = \int_S \det(ds(x)) dr(s(x))$. If $s(r(u)) = u$, then this is a geometric functional. So: geometries $G$ can come from maps from a space $A$ to a space $B$, while fields $F$ can come from maps from $B$ to $A$. The **action integral** $\int_G F$ generalizes the Polyakov action $\int_G \det(dr^T dr) = \int_G |dr|^2$, a case where $F$ and $G$ are dual meaning $s(r(u)) = u$. 

Prototype examples

**Problem:** Compute the line integral of \( F(x, y, z) = [5x^4 + y, 6y^5 + xz, 7z^6 + xy] \) along the path \( r(t) = [\sin(5t), \sin(2t), t^2 / \pi^2] \) from \( t = 0 \) to \( t = 2\pi \).

**Solution:** The field is a gradient field \( df \) with \( f = x^5 + y^6 + z^7 + xyz \). We have \( A = r(0) = (0, 0, 0) \) and \( B = r(2\pi) = (0, 0, 4) \) and \( f(A) = 1 \) and \( f(B) = 4^7 \). The fundamental theorem of line integrals gives \( \int_C \nabla f \cdot dr = f(B) - f(A) = 4^7 \).

**Problem:** Find the line integral of the vector field \( F(x, y) = [x^4 + \sin(x) + y + 5xy, 4x + y^3] \) along the cardioid \( r(t) = (1 + \sin(t))[\cos(t), \sin(t)] \), where \( t \) runs from \( t = 0 \) to \( t = 2\pi \).

**Solution:** We use Green's theorem. Since \( \text{curl}(F) = 3 - 5x \), the line integral is the double integral \( \int_G 3 - 5x \, dx \, dy \). We integrate in polar coordinates and get \( \int_0^{2\pi} \int_0^{1+\sin(t)} (3 - 5r \cos(t)) \, r \, dr \, dt \) which is \( 9\pi / 2 \). One can short cut by noticing that by symmetry \( \int_G (-5x) \, dx \, dy = 0 \), so that the integral is 3 times the area \( \int_0^{2\pi} (1 + \sin(t))^2 / 2 \, dt = 3\pi / 2 \) of the cardioid.

**Problem:** Compute the line integral of \( F(x, y, z) = [x^3 + xy, y, z] \) along the polygonal path \( C \) connecting the points \((0, 0, 0), (2, 0, 0), (2, 1, 0), (0, 1, 0)\).

**Solution:** The path \( C \) bounds a surface \( S : r(u, v) = [u, v, 0] \) parameterized on \( G = \{(x, y) | x \in [0, 2], y \in [0, 1]\} \). By Stokes theorem, the line integral is equal to the flux of \( \text{curl}(F)(x, y, z) = [0, 0, -x] \) through \( S \). The normal vector of \( S \) is \( r_u \times r_v = [1, 0, 0] \times [0, 1, 0] = [0, 0, 1] \) so that \( \int_S \text{curl}(F) \cdot dS = \int_0^2 \int_0^1 [0, 0, -u] \cdot [0, 0, 1] \, dv \, du = \int_0^2 \int_0^1 -u \, dv \, du = -2 \).

**Problem:** Compute the flux of the vector field \( F(x, y, z) = [-x, y, z^2] \) through the boundary \( S \) of the rectangular box \( G = [0, 3] \times [-1, 2] \times [1, 2] \).

**Solution:** By the Gauss theorem, the flux is equal to the triple integral of \( \text{div}(F) = 2z \) over the box: \( \int_0^3 \int_{-1}^2 \int_1^2 2z \, dz \, dy \, dx = (3 - 0)(2 - (-1))(4 - 1) = 27 \).

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