Function spaces and Operators (10.1) 4/25/2002	Math 21b, O. Knill

Homework for Tuesday April 31, 1,2,3,4,5*,6*,7,8,9,10,11*,12* in Section 10.1

LINEAR DIFFERENTIAL EQUATIONS. $Df = f'$ is a linear map on the space of smooth functions C^∞ Also $G_h f = hf$, is a linear map. Composition and addition of linear maps gives new linear maps. For example $Tf = D^2 f + hf$ is a linear map. If T is obtained like this then $Tf = 0$ is called a **linear differential equation**. For linear maps $T = p(D)$ which are polynomials in D, the problem $Tf = 0$ (and more generally the eigenvalue problem $Tf = \lambda f$) are **differential equations with constant coefficients**. The eigenspace to the eigenvalue λ is a linear space. Especially, the kernel of T is a linear space.

FINDING THE KERNEL OF A POLYNOMIAL IN D. How do we find a basis for the kernel of $Tf = f''+2f'+f$? The linear map T can be written as a polynomial in D which means $T = D^2 - D - 2 = (D+1)(D-2)$. The kernel of T contains the kernel of $D-2$ which is one-dimensional and spanned by $f_1 = e^{2x}$. The kernel of $T = (D-2)(D+1)$ also contains the kernel of $D+1$ which is spanned by $f_2 = e^{-x}$. The kernel of T is therefore two dimensional and spanned by e^{2x} and e^{-x}.

THEOREM: If $T = p(D) = D^n + a_{n-1}D^{n-1} + ... + a_1 D + a_0$ on C^∞ then $\dim(\ker(T)) = n$.

PROOF. $T = p(D) = \prod(D - \lambda_j)$, where λ_j are the roots of the polynomial p. The kernel of T contains the kernel of $D - \lambda_j$ which is spanned by $f_j(t) = e^{\lambda_j t}$. In the case when we have a factor $(D-\lambda_j)^k$ of T, then we have to consider the kernel of $(D-\lambda_j)^k$ which is $q(t)e^{\lambda t}$, where q is a polynomial of degree $k-1$. For example, the kernel of $(D-1)^3$ consists of all functions $(a + bt + ct^2)e^t$.

SECOND PROOF. Write this as $ADF = A\dot{F} = 0$, where A is a $n \times n$ matrix and $F = \left[f, \dot{f}, \cdots, f^{(n-1)}\right]^T$, where $f^{(k)} = D^k f$ is the k'th derivative. The linear map $T = AD$ acts on vectors of functions. If all eigenvalues λ_j of A are different (they are the same λ_j as before), then A can be diagonalized. Solving the diagonal case $BD = 0$ is easy. It has a n dimensional kernel of vectors $F = [f_1, \dots, f_n]^T$, where $f_i(t) = t$. If $B = SAS^{-1}$, and F is in the kernel of BD, then SF is in the kernel of AD.

REMARK. The result can be generalized to the case, when a_j are functions of t. Especially, $Tf = g$ has a solution, when T is of the above form. It is important that the highest power D^n is bounded away from 0 for all t. For example $tDf = e^t$ has no solution in C^∞, because we can not integrate e^t/t.

WHY ARE WE INTERESTED IN THE KERNEL?

- Equations $Tf = 0$, where $T = p(D)$ form **linear differential equations with constant coefficients** for which we want to understand the solution space. Such equations are called **homogeneous**. **Solving means finding a basis of the kernel of T.** In the above example, a general solution of $f''+2f'+f = 0$ can be written as $f(t) = a_1 f_1(t) + a_2 f_2(t)$. If we fix two values like $f(0), f'(0)$ or $f(0), f(1)$, the solution is unique.
- If we want to solve $Tf = g$, an **inhomogeneous equation** then T^{-1} is not unique because we have a kernel. If g is in the image of T there is at least one solution f. The general solution is then $f + \ker(T)$. For example, for $T = D^2$, which has C^∞ as its image, we can find a solution to $D^2 f = t^3$ by integrating twice: $f(t) = t^5/20$. The kernel of T consists of all linear functions $at+b$. The general solution to $D^2 = t^3$ is $at + b + t^5/20$. The integration constants parameterize actually the kernel of a linear map.
- In order to find the eigenspace of T to the eigenvalue λ we have to find the kernel of $T - \lambda$.

AN EIGENVALUE PROBLEM. If T is the linear map $Tf = f''$, what are the eigenvalues and eigenvectors?

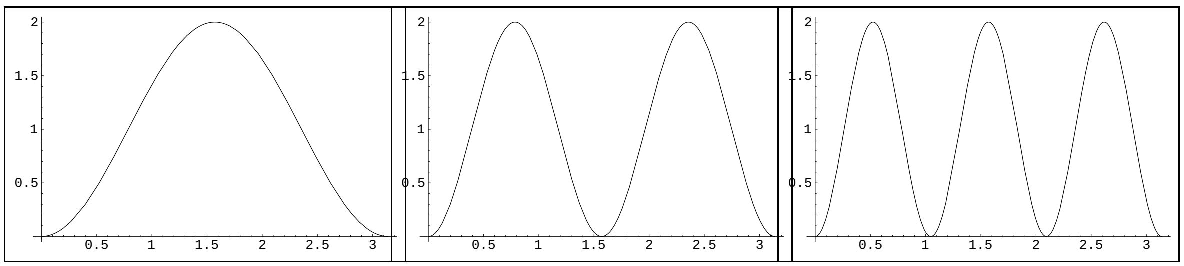
SOLUTION. $Tf = \lambda f$ means $f''(x) = \lambda f(x)$. You remember that the solutions are all of the form $f(x) = a\cos(\lambda t) + b\sin(\lambda t)$. The kernel of D^2 consists of all functions $f(x) = ax + b$. A basis of the kernel are $f_1(x) = 1, f_2(x) = x$. The kernel is two-dimensional.

ON A DIFFERENT SPACE. Let us look at $C^\infty(I)$ consisting of all functions on the interval $I = [0, \pi]$ for which $f(0) = 0$ and $f(\pi) = 0$.
Now, in order that $f(0) = 0$ and $f(\pi) = 0$, we must have $\lambda = n$ and $a = 0$. The linear map T has the eigenfunctions $f_n(x) = \sin(nx)$ to the eigenvalues $\lambda_n = -n^2$. The kernel of T is now trivial because there is no nonzero function f which satisfy $Tf = 0$.

INTERPRETATION. For the eigenvalue problem $Tf = \lambda f$ on $C^\infty(I)$, the numbers λ are the possible **frequencies** of the standing wave which is kept fixed at 0 and π.

QUANTUM MECHANICAL INTERPRETATION. The problem $Tf = \lambda f$ on $C^\infty(I)$ describes the quantum mechanical particle in a box $[0, \pi]$. While $P = i\hbar D$ is the linear map representing the **momentum** of the particle, $H = P^2/2m = -(\hbar^2/2m)D^2$ represents its **kinetic energy**. (In the case of the Hydrogen atom, we had additionally the energy of the y and z direction as well as the potential energy). The functions $\sin(nx)$ are eigenfunctions of $-D^2$ to the eigenvalue $\lambda_n = n^2$, where $n = 1, 2, 3....$ Therefore, $E_n = 2mn^2/\hbar^2$ are the eigenvalues of H.

These are the possible energies of the particle in the box. The quantized appearance of the energies is the origin for the name **"quantum mechanics"**. If a particle is represented by $f_n = \sqrt{2/\pi}\sin(nx)$, which is normalized so that $\int_0^\pi f_n^2 \, dx = 1$, then $f_n^2(x)$ is a probability density. The **probability** to find a particle with energy $2mn^2/\hbar^2$ in an interval $[a, b]$ is $\frac{2}{\pi}\int_a^b \sin^2(nx) \, dx$. Max Planck (on left picture) had been forced to consider a discrete energy spectrum in order to explain the **blackbody radiation**.



Probability distribution in $n = 1$ Probability distribution in $n = 2$ Probability distribution in $n = 3$

MOTIVATION: WAVES. If we bend a string located on the graph of a function $x \mapsto T(x)$ on $[0, \pi]$ satisfying $T(0) = T(\pi) = 0$, then the force $F(x)$ which pulls it back at the point x is proportional to T''. The string $T(x, t)$ satisfies $\ddot{T}(x, t) = c^2 T''(x, t)$, where c is a constant. If we write $T(x, t) = u(x)v(t)$, then $\ddot{T} = \ddot{v}u$ and $T'' = vu''$. The equation becomes now $\ddot{v}u = c^2 vu''$ or $\ddot{v}/(c^2 v) = u''/u$. Because the left hand side depends only on t and the right hand side only on x, we have $\ddot{v}/(c^2 v) = u''/u = -n^2 = const$. The right equation is an eigenvalue problem $u'' = \lambda u$ which has solutions for $\lambda = -n^2$. The eigenvectors are $u_n(x) = \sin(nx)$. Now, $v_n(t) = \exp(inct)$ solves $\ddot{v} = -c^2 n^2 v$ so that $T_n(x, t) = u_n(x)v_n(t) = \sin(nx)\exp(inct)$ are solutions of the wave equation. General solutions can be obtained by taking **superpositions of these waves** $T(x, t) = \sum_n c_n \sin(nx)\exp(inct)$. The coefficients $c_n = a_n + ib_n$ are obtained from $T(x, 0) = \sum_n a_n \sin(nx)$ and $\dot{T}(x, 0) = \sum_n b_n nc \sin(nx)$. These are **Fourier series** which we will look at in the next class.