COMPLEX LINEAR 1D CASE. \( \dot{x} = \lambda x \) for \( \lambda = a+i b \) has solution \( x(t) = e^{at} e^{ibt} x(0) \) and norm \( \|x(t)\| = e^{at}\|x(0)\| \).

OSCILLATOR: The system \( \ddot{x} = -\lambda x \) has the solution \( x(t) = \cos(\sqrt{\lambda}t) x(0) + \sin(\sqrt{\lambda}t) \dot{x}(0)/\sqrt{\lambda} \).

DERIVATION. \( \dot{x} = y, \dot{y} = -\lambda x \) and in matrix form as
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
0 & -1 \\
\lambda & 0
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = A \begin{bmatrix}
x \\
y
\end{bmatrix}
\]
and because \( A \) has eigenvalues \( \pm i \sqrt{\lambda} \), the new coordinates move as \( a(t) = e^{i \sqrt{\lambda}t} a(0) \) and \( b(t) = e^{-i \sqrt{\lambda}t} b(0) \).

Writing this in the original coodinates
\[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} = S \begin{bmatrix}
a(t) \\
b(t)
\end{bmatrix}
\]
and fixing the constants gives \( x(t), y(t) \).

EXAMPLE. THE SPINNER. The spinner is a rigid body attached to a spring aligned around the z-axes. The body can rotate around the z-axes and bounce up and down. The two motions are coupled in the following way: when the spinner winds up in the same direction as the spring, the spring gets tightened and the body gets a lift. If the spinner winds up to the other direction, the spring becomes more relaxed and the body is lowered.

SETTING UP THE DIFFERENTIAL EQUATION.
\( x \) is the angle and \( y \) the height of the body. We put the coordinate system so that \( y = 0 \) is the point, where the body stays at rest if \( x = 0 \). We assume that if the spring is wound up with an angle \( x \), this produces an upwards force \( x \) and a momentum force \(-3 \times x\). We furthermore assume that if the body is at position \( y \), then this produces a momentum \( y \) onto the body and an upwards force \( y \).

The differential equations
\[
\begin{align*}
\dot{x} &= -3x + y \\
\dot{y} &= -y + x
\end{align*}
\]
can be written as \( \ddot{v} = Av = \begin{bmatrix}
-3 & 1 \\
1 & -1
\end{bmatrix} v \).

FINDING GOOD COORDINATES \( w = S^{-1} v \) is obtained with getting the eigenvalues and eigenvectors of \( A \): \( \lambda_1 = -2 - \sqrt{2} \), \( \lambda_2 = -2 + \sqrt{2} \)
\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix}
-1 - \sqrt{2} \\
1
\end{bmatrix}, \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix}
-1 + \sqrt{2} \\
1
\end{bmatrix}
\]
so that
\[
S = \begin{bmatrix}
-1 - \sqrt{2} & -1 + \sqrt{2} \\
1 & 1
\end{bmatrix}.
\]

SOLVE THE SYSTEM \( \ddot{a} = \lambda_1 a, \ddot{b} = \lambda_2 b \) IN THE GOOD COORDINATES
\[
\begin{bmatrix}
a(t) \\
b(t)
\end{bmatrix} = S^{-1} \begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix}.
\]

THE SOLUTION IN THE ORIGINAL COORDINATES.
\[
S \begin{bmatrix}
a(t) \\
b(t)
\end{bmatrix}.
\]
At \( t = 0 \) we know \( x(0), y(0), \dot{x}(0), \dot{y}(0) \). This fixes the constants in \( x(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \).

ASYMPTOTIC STABILITY.
A linear system \( \ddot{x} = Ax \) in the 2D plane is asymptotically stable if and only if \( \det(A) > 0 \) and \( \text{tr}(A) < 0 \).

PROOF. If the eigenvalues \( \lambda_1, \lambda_2 \) of \( A \) are real then both being negative is equivalent with \( \lambda_1 \lambda_2 = \det(A) > 0 \) and \( \text{tr}(A) = \lambda_1 + \lambda_2 < 0 \). If \( \lambda_1 = a + ib, \lambda_2 = a - ib \), then a negative \( a \) is equivalent to \( \lambda_1 + \lambda_2 = 2a < 0 \) and \( \lambda_1 \lambda_2 = a^2 + b^2 > 0 \).
ASYMPTOTIC STABILITY COMPARISON OF DISCRETE AND CONTINUOUS SITUATION.
The trace and the determinant are independent of the basis, they can be computed fast, and are real if
$A$ is real. It is therefore convenient to determine the region in the $\text{tr} - \text{det}$-plane, where continuous or
discrete dynamical systems are asymptotically stable. While the continuous dynamical system is related
to a discrete system, it is important not to mix these two situations up.

<table>
<thead>
<tr>
<th>Continuous dynamical system.</th>
<th>Discrete dynamical system.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stability of $\dot{x} = Ax$ ($x(t+1) = e^{A}x(t)$).</td>
<td>Stability of $x(t+1) = Ax$</td>
</tr>
</tbody>
</table>

- Stability in $\det(A) > 0, \text{tr}(A) > 0$
- Stability if $\text{Re}(\lambda_1) < 0, \text{Re}(\lambda_2) < 0.$

- Stability in $|\text{tr}(A)| - 1 < \det(A) < 1$
- Stability if $|\lambda_1| < 1, |\lambda_2| < 1.$

PHASE PORTRAITS. (In two dimensions we can plot the vector field, draw some trajectories)

- $\lambda_1 < 0, \lambda_2 < 0,$
i.e $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

- $\lambda_1 > 0, \lambda_2 > 0,$
i.e $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

- $\lambda_1 = 0, \lambda_2 = 0,$
i.e $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

- $\lambda_1 < 0, \lambda_2 > 0,$
i.e $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

- $\lambda_1 = 0, \lambda_2 < 0,$
i.e $A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$

- $\lambda_1 = a + ib, a < 0, \lambda_2 = a - ib,$
i.e $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

- $\lambda_1 = a + ib, a > 0, \lambda_2 = a - ib,$
i.e $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

- $\lambda_1 = ib, a < 0, \lambda_2 = -ib,$
i.e $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$