

Homework for Tuesday, April 2: 6.3 2,4*,12,18*,22,31*

AIM. We want explicit formulas for the inverse of a matrix A or the solution x of a linear equation $Ax = b$. While writing this in terms of determinants is **not** the most efficient way to compute these things, such formulas are useful for example when having parameters in the matrix. A symbolic algebra program can then for example give explicit formulas for the dependence of the solution x on external parameters and allow theoretical predictions.

REMINDERS.

- An orthonormal matrix satisfies $Q^T Q = 1$. From this we get $\det(Q)^2 = 1$ and so $|\det(Q)| = 1$.
- The determinant of a tridiagonal matrix is the product of the diagonal entries.
- Every matrix can be written as $A = QR$, where Q is orthogonal and R is upper triangular.
- The image of the unit cube under a linear map $x \mapsto Ax$ is a parallelepiped E_n spanned by the column vectors v_1, \dots, v_n of A .

VOLUME OF A PARALLELEPIPED. A j -dimensional parallelepiped E_j spanned by vectors v_1, \dots, v_j has a $j-1$ dimensional parallelepiped E_{j-1} in the base. E_{j-1} is contained in the vector space V_{j-1} spanned by v_1, \dots, v_{j-1} . The opposite "face" is in distance $\|u_j\| = \|v_j - \text{proj}_{V_{j-1}} v_j\|$. The volume $\text{vol}(E_j)$ satisfies $\text{vol}(E_j) = \text{vol}(E_{j-1})\|u_j\|$.

ORIENTATION. Determinants allow us to **define** the orientation of n vectors in n -dimensional space, (where we don't have a "right hand rule" in general ...). Just look at the matrix A with column vectors v_j and define the orientation as the sign of $\det(A)$. In three dimensions, this agrees with the right hand rule: if v_1 is the thumb, v_2 is the pointing finger and v_3 is the middle finger, then their orientation is positive.

DETERMINANT AND VOLUME. The absolute value of the determinant of a $n \times n$ matrix A is the volume of the n -dimensional parallelepiped E_n spanned by the column vectors v_j of A .

Proof. Use the QR decomposition $A = QR$, where Q is orthogonal and R is upper triangular. From $QQ^T = 1$, we get $1 = \det(Q)\det(Q^T) = \det(Q)^2$ see that $|\det(Q)| = 1$. Therefore, $\det(A) = \pm\det(R)$. The determinant of R is the product of the $\|u_i\| = \|v_i - \text{proj}_{V_{j-1}} v_i\|$ which was the distance from v_i to V_{j-1} . The volume $\text{vol}(E_j)$ of a j -dimensional parallelepiped E_j with base E_{j-1} in V_{j-1} and height $\|u_i\|$ is $\text{vol}(E_{j-1})\|u_j\|$. Inductively $\text{vol}(E_j) = \|u_j\|\text{vol}(E_{j-1})$ and therefore $\text{vol}(E_n) = \prod_{j=1}^n \|u_j\| = \det(R)$.

MORE GENERALLY: The volume of a k dimensional parallelepiped defined by the vectors v_1, \dots, v_k is $\sqrt{\det(A^T A)}$ because $A^T A = (QR)^T (QR) = R^T R$ and $\det(R^T R) = \det(R)^2 = (\prod_{j=1}^n \|u_j\|)^2$.

CHANGE OF VARIABLES. (Relation with multi-variable calculus) If $x \mapsto y = u(x)$ is a change of variable, then the matrix $Du(x)$ is the linearisation of the map near x and $\|dy\| = |\det(Du(x))| \cdot \|dx\|$. This leads to the change of variable formula

$$\int_S f(x) dx = \int_{u(S)} f(y) |\det(Du^{-1}(y))| dy$$

(see example 1 below). If u is a map from \mathbf{R}^m to \mathbf{R}^n , where $m \leq n$, then the expansion factor is $\sqrt{\det(A^T A)}$, where $A = Du$. (See example 2 below).

Example 1: (1 dim) if $x = u^{-1}(y) = \sin(y)$, $dx = \cos(y) dy$. $\int_0^1 \sqrt{1-x^2} dx = \int_0^{\pi/2} \sqrt{(1-\sin^2(y))} \cos(y) dy = \int_0^{\pi/2} \cos^2(y) dy = \pi/4$.

Example 2: If $u(s, t) = (x(s, t), y(s, t), z(s, t))$ is a surface, then $A = Du(s, t)$ is a 3×2 matrix with column vectors $X = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix}$ $Y = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}$ (which are tangent vectors to the surface). Now $A^T A = \begin{bmatrix} X \cdot X & X \cdot Y \\ X \cdot Y & Y \cdot Y \end{bmatrix}$ whose determinant is $\|X\|^2\|Y\|^2 - \|X \cdot Y\|^2 = \|X\|^2\|Y\|^2(1 - \cos(\phi)^2) = \|X\|^2\|Y\|^2 \sin(\phi)^2 = \|X \times Y\|^2$. The expansion factor is $\sqrt{\det(A^T A)} = \|X \times Y\|$.

CRAMER'S RULE. This is an explicit formula for the solution of $Ax = b$. If A_i denotes the matrix, where the column v_i is replaced by b , then

$$x_i = \det(A_i) / \det(A).$$

Proof. $\det(A_i) = \det([v_1, \dots, b, \dots, v_n]) = \det([v_1, \dots, (Ax), \dots, v_n]) = \det([v_1, \dots, \sum_i x_i v_i, \dots, v_n]) = x_i \det([v_1, \dots, v_n]) = x_i \det(A)$.



GABRIEL CRAMER. (1704-1752). Born in Geneva (Switzerland), he worked on geometry and analysis. He died during a trip to France, where he wanted to start his retirement.

Cramer used the rule named after him in a book "Introduction à l'analyse des lignes courbes algébrique", where he solved like this a system of equations with 5 unknowns. According to a short biography of Cramer by J J O'Connor and E F Robertson, the rule had however been used already before by other mathematicians.

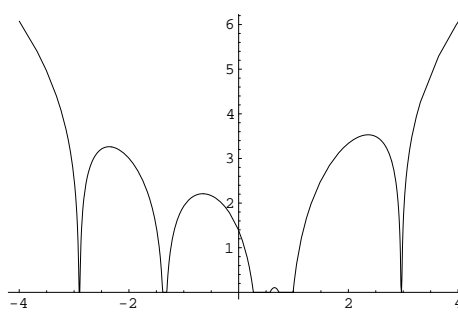
WHY IS CRAMERS RULE INTERESTING? Determining x with these formulas is slower than with Gaussian elimination: a determinant calculation needs n^3 steps so that n^4 calculations are needed for the inverse via Cramer's rule. (Compare n^3 with Gaussian elimination).

The rule is still important however because if A or b depends on a parameter λ , and we want to see how x depends on the parameter λ one can find explicit formulas for $(d/d\lambda)x_i(\lambda)$. Cramers rule tells for example that the solution can depend in a sensitive way on parameters if the determinant is small. (Look at scissors).

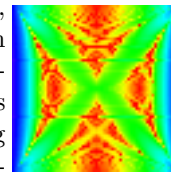
EXAMPLE. In solid state physics, one is interested in the $\det(L - E)$, where

$$L = \begin{bmatrix} \lambda \cos(\alpha) & 1 & 0 & \cdot & 0 & 1 \\ 1 & \lambda \cos(2\alpha) & 1 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & \cdot & \cdot & 1 & \lambda \cos((n-1)\alpha) & 1 \\ 1 & 0 & \cdot & 0 & 1 & \lambda \cos(n\alpha) \end{bmatrix}$$

describes an electron in a periodic crystal, E is the energy and $\alpha = 2\pi/n$. The electron can move (as a Bloch wave) whenever the determinant is negative. These intervals form the **spectrum** of the matrix. A physicist is interested for example in the dependence of the spectrum on the parameter λ or E .



The graph to the left shows the function $E \mapsto \log(|\det(L-E)|)$ in the case $\lambda = 2$ and $n = 5$. In the energy intervals, where this function is zero, the electron can move, otherwise the crystal is an insulator. The picture to the right shows the spectrum of the crystal depending on α . It is called the "Hofstadter butterfly" (popularized in "Gödel, Escher Bach" by D. Hofstadter).



THE INVERSE OF A MATRIX. Because the columns of A^{-1} are solutions of $Ax = e_i$ with $e_i =$

$$\begin{bmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{bmatrix},$$

Cramers rule together with the Lagrange expansion gives $[A^{-1}]_{ij} = e_j \cdot A^{-1} e_i = e_j \cdot x_j = (-1)^{i+j} \det(A_{ji}) / \det(A)$.

$$[A^{(-1)}]_{ij} = (-1)^{i+j} \det(A_{ji}) / \det(A)$$

The matrix $B_{ij} = (-1)^{i+j} \det(A_{ji})$ is called the **classical adjoint** of A . NOTE the change $ij \rightarrow ji$. **Don't** confuse the classical adjoint with the **transpose** A^T which is sometimes also called the **adjoint**.